Supplemental Material

This Supplemental Material contains a detailed presentation of the analysis whose results are presented in the main text.

In the first sections of this Supplemental Material (from A to D) we present the derivation of the quenched constrained complexity. The derivation follows closely the one presented in Ref. [1] for a similar setup, and we refer to that work for results that extend straightforwardly to this case. In Sec. E we show that, at the saddle point, the quenched complexity reproduces the annealed result presented in the main text. In Sec. F we derive the stability of the typical stationary points counted by the complexity, through the analysis of the statistical properties of their Hessian. In Sec. G we report some additional results on the complexity. Sec. H contains the calculation of the zero-temperature Franz-Parisi potential.

A. Replicated Kac-Rice formula

This complexity in Eq. (2) of the main text is *quenched* since the disorder average is performed over an intensive quantity (the logarithm of the number $\mathcal{N}_{\boldsymbol{\sigma}^0}$), rather that over the number itself. The *annealed* version of the complexity is obtained exchanging the disordered average with the logarithm, and can be computed expressing the first moment of $\mathcal{N}_{\boldsymbol{\sigma}^0}$ by means of the the Kac-Rice formula. This reads (see also Ref. [2]):

$$\langle \mathcal{N}_{\boldsymbol{\sigma}^0} \rangle_0 = \int d\boldsymbol{\sigma}\, \delta\big(\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}^0 - q\big)\, \langle |\det \mathcal{H}[\boldsymbol{\sigma}]| \rangle_0\, p_{\boldsymbol{\sigma}|\boldsymbol{\sigma}^0}(\mathbf{0},\epsilon), \tag{1}$$

where the integration is over configurations $\boldsymbol{\sigma}$ on the unit sphere, and $p_{\boldsymbol{\sigma}|\boldsymbol{\sigma}^0}(\mathbf{0},\epsilon)$ is the joint density function of the gradient and field $(\mathbf{g}[\boldsymbol{\sigma}], h[\boldsymbol{\sigma}])$ evaluated at $(\mathbf{0}, \sqrt{2N}\epsilon)$, conditioned to $\boldsymbol{\sigma}^0$. The first moment of $\mathcal{N}_{\boldsymbol{\sigma}^0}$ is thus formed by three terms: the joint distribution gives the probability that $\boldsymbol{\sigma}$ is a stationary point and accounts for its correlations with $\boldsymbol{\sigma}^0$, the expectation value of the determinant counts the multiplicity of stationary points in a given level set of the landscape, and the integration over the volume at fixed overlap q accounts for the phase space available to them. Three analogous contributions appear also in the calculation of the quenched complexity. To perform the averages of the logarithm, we exploit the replica trick:

$$\Sigma(\epsilon, q|\epsilon_0) = \lim_{N\to\infty} \lim_{n\to 0} \frac{M_n(\epsilon, q|\epsilon_0) - 1}{Nn}, \tag{2}$$

where

$$M_n(\epsilon, q|\epsilon_0) \equiv \Big\langle \mathcal{N}^n_{\boldsymbol{\sigma}^0}(\epsilon, q|\epsilon_0) \Big| \begin{Bmatrix} \mathbf{g}[\boldsymbol{\sigma}^0]=0, \\ h[\boldsymbol{\sigma}^0]=\sqrt{2N}\epsilon_0 \end{Bmatrix} \Big\rangle \tag{3}$$

is the expression for the higher moments of $\mathcal{N}_{\boldsymbol{\sigma}^0}$, which can be obtained by replicating the Kac-Rice formula for the first moment. This involves introducing n configurations $\boldsymbol{\sigma}^a$, $a = 1, \cdots, n$ (which we henceforth refer to as *replicas*), all at fixed overlap q with $\boldsymbol{\sigma}^0$. For all the $n+1$ points labeled [3] by $\alpha = 0, 1, \cdots, n$ we define the gradient vectors $\mathbf{g}^\alpha \equiv \mathbf{g}[\boldsymbol{\sigma}^\alpha]$, the Hessian $\mathcal{H}^\alpha \equiv \mathcal{H}[\boldsymbol{\sigma}^\alpha]$, and the value of the rescaled energy functional $h^\alpha \equiv h[\boldsymbol{\sigma}^\alpha]$ defined in the main text. We denote with $\vec{\mathbf{g}} = (\mathbf{g}^1, \cdots, \mathbf{g}^n)$ the $(N-1)n$-dimensional vector collecting the gradients of the n replicas, and with $\vec{h} = (h^1, \cdots, h^n)$ the collection of the n functionals h^a. We let $p_{\vec{\boldsymbol{\sigma}}|\boldsymbol{\sigma}^0}$ be the joint density function of the gradients $\vec{\mathbf{g}}$ and fields $\vec{h}$, induced by the distribution of the couplings and conditioned to $\mathbf{g}^0 = \mathbf{0}$ and $h^0 = \sqrt{2N}\epsilon_0$. With this notation, the replicated version of the Kac-Rice formula reads:

$$M_n = \int \prod_{a=1}^{n} d\boldsymbol{\sigma}^a\, \delta\left(\boldsymbol{\sigma}^a\cdot\boldsymbol{\sigma}^0 - q\right) \mathcal{E}_{\vec{\boldsymbol{\sigma}}|\boldsymbol{\sigma}^0}(\epsilon)\, p_{\vec{\boldsymbol{\sigma}}|\boldsymbol{\sigma}^0}(\mathbf{0},\epsilon), \tag{4}$$

where the integration is over configurations $\boldsymbol{\sigma}^a$ constrained to be in the unit sphere, at overlap q with the fixed minimum $\boldsymbol{\sigma}^0$. In (4), $p_{\vec{\boldsymbol{\sigma}}|\boldsymbol{\sigma}^0}(\mathbf{0},\epsilon)$ is a shorthand notation for the joint density evaluated at $\vec{\mathbf{g}} = 0$ and $h^a = \sqrt{2N}\epsilon$ for any $a = 1, \cdots, n$, while

$$\mathcal{E}_{\vec{\boldsymbol{\sigma}}|\boldsymbol{\sigma}^0}(\epsilon) = \left\langle \left(\prod_{a=1}^{n} |\det \mathcal{H}^a| \right) \Big| \begin{Bmatrix} h^a=\sqrt{2N}\epsilon, h^0=\sqrt{2N}\epsilon_0 \\ \mathbf{g}^a=\mathbf{0}\ \forall a=0,\ldots,n \end{Bmatrix} \right\rangle \tag{5}$$

denotes the expectation value of the product of the determinants of the Hessians of all replicas, conditioned on each $\boldsymbol{\sigma}^a$ being a stationary point with rescaled energy $\sqrt{2N}\epsilon$ and overlap q with the stationary point $\boldsymbol{\sigma}^0$.

To extract the leading order in N of (4), we need to characterize the joint distribution of the energy, gradient and Hessian fields at the points $\boldsymbol{\sigma}^a$, conditioned to the presence of $\boldsymbol{\sigma}^0$. This involves choosing a set of $n+1$ orthonormal bases $\mathcal{B}[\boldsymbol{\sigma}^\alpha] = \left\{\mathbf{e}^\alpha_1, \cdots, \mathbf{e}^\alpha_{N-1}\right\}$ in the tangent planes at each $\boldsymbol{\sigma}^\alpha$, and computing the averages and covariances of all the fields components with respect to these bases. As it follows from the isotropy of the covariance field of the p-spin Hamiltonian, the resulting correlations depend only on the scalar products $\mathbf{e}^\alpha_i \cdot \mathbf{e}^\beta_j$ and $\mathbf{e}^\alpha_i \cdot \boldsymbol{\sigma}^\beta$, see the following section for the explicit expressions. If the bases $\mathcal{B}[\boldsymbol{\sigma}^\alpha]$ are chosen suitably, the joint distribution of all fields components can be parametrized only in terms of the mutual overlaps $q_{\alpha\beta} = \boldsymbol{\sigma}^\alpha \cdot \boldsymbol{\sigma}^\beta$ between all configurations (included the fixed overlap $q_{0a} = q$ with $\boldsymbol{\sigma}^0$). This allows to re-write (4) as:

$$M_n = \int \prod_{a<b=1}^{n} dq_{ab}\, e^{N S_n(\epsilon, \hat{Q}|\epsilon_0) + o(Nn)}, \tag{6}$$

where the leading-order term at the exponent depends on the $\boldsymbol{\sigma}^a$ only through the $(n+1)\times(n+1)$ overlap matrix

$\hat{Q}$ with components:

$$Q_{\alpha\beta} = \delta_{\alpha\beta} + (1-\delta_{\alpha\beta})\left[q_{\alpha\beta} + (\delta_{\alpha 0} + \delta_{\beta 0})(q - q_{\alpha\beta})\right]. \quad (7)$$

The quenched complexity is determined by the linear term in n of $S_n(\epsilon, \hat{Q}|\epsilon_0)$. We thus set:

$$S_n(\epsilon, \hat{Q}|\epsilon_0) = n\Sigma(\epsilon, \hat{Q}|\epsilon_0) + O(n^2), \quad (8)$$

and derive in Sec. D the explicit expression of $\Sigma(\epsilon, \hat{Q}|\epsilon_0)$. This is obtained within the ansatz $q_{ab} \equiv q_1$, which corresponds to assuming a 1RSB structure of the landscape in the vicinity of the fixed minimum. The calculation is concluded by performing the integral over q_1 with the saddle point method, see Sec. E.

B. Covariances of fields and choice of basis vectors

To evaluate explicitly Eq. (4), we need to characterize of the joint distribution of the fields $\mathcal{H}^a$, $\mathbf{g}^a$ and h^a.
We remind that $\mathbf{g}^a$ and $\mathcal{H}^a$ denote the *Riemannian* gradient and Hessian fields, which account for the spherical constraint and which lie in the tangent plane to the sphere at each $\boldsymbol{\sigma}^a$. For simplicity, for each $\alpha = 0, 1, \cdots, n$ we introduce also the gradients $\boldsymbol{\nabla} h^\alpha \equiv \boldsymbol{\nabla} h[\boldsymbol{\sigma}^\alpha]$ and Hessian $\boldsymbol{\nabla}^2 h^\alpha \equiv \boldsymbol{\nabla}^2 h[\boldsymbol{\sigma}^\alpha]$ of the rescaled energy functional *extended* to the whole N-dimensional space, and determine the covariances between their components along arbitrary directions in this space, given by some N-dimensional unit vectors $\mathbf{v}_i$. The correlations of the components $\mathbf{g}^\alpha$ and $\mathcal{H}^\alpha$ of the Riemannian gradients and Hessians are easily determined choosing $\mathbf{v}_i \to \mathbf{e}^\alpha_\beta$ to be vectors on the tangent plane at the various $\boldsymbol{\sigma}^\alpha$; indeed, $\mathbf{g}^\alpha$ is an $(N-1)$-dimensional vector with components $g^\alpha_\beta = \boldsymbol{\nabla} h^\alpha \cdot \mathbf{e}^\alpha_\beta$, obtained from $\boldsymbol{\nabla} h^\alpha$ by a projection onto the tangent plane. Similarly, the Riemannian Hessian $\mathcal{H}^\alpha$ is an $(N-1)\times(N-1)$ matrix whose components are related to the ones of $\boldsymbol{\nabla}^2 h^\alpha$ by:

$$\mathcal{H}^\alpha_{\beta\gamma} = \mathbf{e}^\alpha_\beta \cdot \left(\boldsymbol{\nabla}^2 h^\alpha - (\boldsymbol{\nabla} h^\alpha \cdot \boldsymbol{\sigma}^\alpha)\,\hat{1}\right) \cdot \mathbf{e}^\alpha_\gamma. \quad (9)$$

For arbitrary $\mathbf{v}_i$ it holds

$$\begin{aligned}
&\left\langle (\boldsymbol{\nabla} h^\alpha \cdot \mathbf{v}_1)\, h^\beta \right\rangle = p(\boldsymbol{\sigma}^\alpha \cdot \boldsymbol{\sigma}^\beta)^{p-1} \left(\mathbf{v}_1 \cdot \boldsymbol{\sigma}^\beta\right), \\
&\left\langle \left(\mathbf{v}_1 \cdot \boldsymbol{\nabla}^2 h^\alpha \cdot \mathbf{v}_2\right) h^\beta \right\rangle = p(p-1)(\boldsymbol{\sigma}^\alpha \cdot \boldsymbol{\sigma}^\beta)^{p-2}(\mathbf{v}_1 \cdot \boldsymbol{\sigma}^\beta)(\mathbf{v}_2 \cdot \boldsymbol{\sigma}^\beta),
\end{aligned} \quad (10)$$

while the covariances between the gradient components read:

$$\begin{aligned}
&\left\langle (\boldsymbol{\nabla} h^\alpha \cdot \mathbf{v}_1)\left(\boldsymbol{\nabla} h^\beta \cdot \mathbf{v}_2\right) \right\rangle = p(\boldsymbol{\sigma}^\alpha \cdot \boldsymbol{\sigma}^\beta)^{p-1}(\mathbf{v}_1 \cdot \mathbf{v}_2) + \\
&p(p-1)(\boldsymbol{\sigma}^\alpha \cdot \boldsymbol{\sigma}^\beta)^{p-2}(\mathbf{v}_2 \cdot \boldsymbol{\sigma}^\alpha)\left(\mathbf{v}_1 \cdot \boldsymbol{\sigma}^\beta\right).
\end{aligned} \quad (11)$$

For what concerns the Hessians, one gets:

$$\begin{aligned}
&\left\langle \left(\mathbf{v}_1 \cdot \boldsymbol{\nabla}^2 h^\alpha \cdot \mathbf{v}_2\right)\left(\mathbf{v}_3 \cdot \boldsymbol{\nabla}^2 h^\beta \cdot \mathbf{v}_4\right) \right\rangle = \\
&\frac{p!(\boldsymbol{\sigma}^\alpha \cdot \boldsymbol{\sigma}^\beta)^{p-4}}{(p-4)!}(\mathbf{v}_1 \cdot \boldsymbol{\sigma}^\beta)(\mathbf{v}_2 \cdot \boldsymbol{\sigma}^\beta)(\mathbf{v}_3 \cdot \boldsymbol{\sigma}^\alpha)(\mathbf{v}_4 \cdot \boldsymbol{\sigma}^\alpha) + \\
&\frac{p!}{(p-3)!}(\boldsymbol{\sigma}^\alpha \cdot \boldsymbol{\sigma}^\beta)^{p-3}(\mathbf{v}_1 \cdot \mathbf{v}_4)(\mathbf{v}_2 \cdot \boldsymbol{\sigma}^\beta)(\mathbf{v}_3 \cdot \boldsymbol{\sigma}^\alpha) + \\
&\frac{p!}{(p-3)!}(\boldsymbol{\sigma}^\alpha \cdot \boldsymbol{\sigma}^\beta)^{p-3}(\mathbf{v}_2 \cdot \mathbf{v}_4)(\mathbf{v}_1 \cdot \boldsymbol{\sigma}^\beta)(\mathbf{v}_3 \cdot \boldsymbol{\sigma}^\alpha) + \\
&\frac{p!}{(p-3)!}(\boldsymbol{\sigma}^\alpha \cdot \boldsymbol{\sigma}^\beta)^{p-3}(\mathbf{v}_1 \cdot \mathbf{v}_3)(\mathbf{v}_2 \cdot \boldsymbol{\sigma}^\beta)(\mathbf{v}_4 \cdot \boldsymbol{\sigma}^\alpha) + \\
&\frac{p!}{(p-3)!}(\boldsymbol{\sigma}^\alpha \cdot \boldsymbol{\sigma}^\beta)^{p-3}(\mathbf{v}_2 \cdot \mathbf{v}_3)(\mathbf{v}_1 \cdot \boldsymbol{\sigma}^\beta)(\mathbf{v}_4 \cdot \boldsymbol{\sigma}) + \\
&\frac{p!(\boldsymbol{\sigma}^\alpha \cdot \boldsymbol{\sigma}^\beta)^{p-2}}{(p-2)!}\left[(\mathbf{v}_1 \cdot \mathbf{v}_3)(\mathbf{v}_2 \cdot \mathbf{v}_4) + (\mathbf{v}_1 \cdot \mathbf{v}_4)(\mathbf{v}_2 \cdot \mathbf{v}_3)\right].
\end{aligned} \quad (12)$$

Finally, the correlations between Hessians and gradients read:

$$\begin{aligned}
&\left\langle \left(\mathbf{v}_1 \cdot \boldsymbol{\nabla}^2 h^\alpha \cdot \mathbf{v}_2\right)\left(\boldsymbol{\nabla} h^\beta \cdot \mathbf{v}_3\right) \right\rangle = \\
&p(p-1)(p-2)(\boldsymbol{\sigma}^\alpha \cdot \boldsymbol{\sigma}^\beta)^{p-3}(\mathbf{v}_1 \cdot \boldsymbol{\sigma}^\beta)(\mathbf{v}_2 \cdot \boldsymbol{\sigma}^\beta)(\mathbf{v}_3 \cdot \boldsymbol{\sigma}^\alpha) + \\
&p(p-1)(\boldsymbol{\sigma}^\alpha \cdot \boldsymbol{\sigma}^\beta)^{p-2}(\mathbf{v}_1 \cdot \mathbf{v}_3)(\mathbf{v}_2 \cdot \boldsymbol{\sigma}^\beta) + \\
&p(p-1)(\boldsymbol{\sigma}^\alpha \cdot \boldsymbol{\sigma}^\beta)^{p-2}(\mathbf{v}_2 \cdot \mathbf{v}_3)(\mathbf{v}_1 \cdot \boldsymbol{\sigma}^\beta).
\end{aligned} \quad (13)$$

The covariances of the components along all the directions $\mathbf{v}_i$ that are orthogonal to the $\boldsymbol{\sigma}^\alpha$ with $\alpha = 0, \cdots, n$ have a simple form. We thus choose the bases $\mathcal{B}[\boldsymbol{\sigma}^\alpha]$ in each tangent plane in such a way that the last n vectors $\mathbf{e}^\alpha_{N-n-1}, \cdots, \mathbf{e}^\alpha_{N-1}$, together with the normal direction $\boldsymbol{\sigma}^\alpha$, span the $(n+1)$-dimensional subspace $S \equiv \text{span}\left\{\boldsymbol{\sigma}^0, \boldsymbol{\sigma}^1, \cdots, \boldsymbol{\sigma}^n\right\}$, while the remaining $N-1-n$ vectors span the orthogonal subspace $S^\perp$. Since for each $\alpha = 0, 1, \cdots, n$ the vectors generating $S^\perp$ are automatically orthogonal to $\boldsymbol{\sigma}^\alpha$, they can be chosen to be equal in each tangent plane, independently of α. We denote these vectors simply with $\mathbf{e}_i$ for $i = 1, \cdots, N-1-n$. On the contrary, the n vectors $\mathbf{e}^\alpha_{N-1-n}, \cdots, \mathbf{e}^\alpha_{N-n}$ have to be chosen in an α-dependent way, since they have to be orthogonal to the normal direction $\boldsymbol{\sigma}^\alpha$. Notice that the sets $\tilde{\mathcal{B}}[\boldsymbol{\sigma}^\alpha] \equiv \{\mathcal{B}[\boldsymbol{\sigma}^\alpha], \boldsymbol{\sigma}^\alpha\}$ are orthonormal bases of the full N-dimensional space in which the sphere is embedded, which can be mapped into each others by unitary transformations.
The components of the gradients and Hessians along the first $M = N-1-n$ directions in each tangent plane are uncorrelated with each others, and are uncorrelated with the energy fields of all replicas. They satisfy:

$$\left\langle (\boldsymbol{\nabla} h^\alpha \cdot \mathbf{e}_i)\left(\boldsymbol{\nabla} h^\beta \cdot \mathbf{e}_j\right) \right\rangle = p(\boldsymbol{\sigma}^\alpha \cdot \boldsymbol{\sigma}^\beta)^{p-1}\delta_{ij}, \quad (14)$$

and

$$\left\langle \left(\mathbf{e}_i\cdot\boldsymbol{\nabla}^2 h^\alpha\cdot\mathbf{e}_j\right)\left(\mathbf{e}_k\cdot\boldsymbol{\nabla}^2 h^\beta\cdot\mathbf{e}_l\right)\right\rangle = \frac{p!(\boldsymbol{\sigma}^\alpha\cdot\boldsymbol{\sigma}^\beta)^{p-2}}{(p-2)!}\times \\ \times\left(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}\right). \tag{15}$$

The covariances of the remaining components along the directions $\mathbf{e}_i^a$ depend instead on the particular choice of these basis vectors in each tangent plane. However, since these vectors span the subspace S, they can be expressed as linear combinations of the $\boldsymbol{\sigma}^\alpha$, implying that their covariances are functions only of the overlaps between replicas, and can thus be parametrized by q and $q_{ab}=\boldsymbol{\sigma}^a\cdot\boldsymbol{\sigma}^b$: the joint and conditional distributions of the rescaled energy field, its gradient and Hessian thus depend only on these parameters, implying that the action in (6) in turns depends only on these parameters.

To perform explicit calculations in the following, we introduce one specific choice of these basis vectors $\mathbf{e}_i^a$ in each tangent plane. For the first replica $\boldsymbol{\sigma}^1$, we set:

$$\mathbf{e}^1_{M+k}=\frac{1}{\sqrt{(k+1)k(1-q_1)}}\left(\sum_{b=2}^{k+1}\boldsymbol{\sigma}^b-k\boldsymbol{\sigma}^{k+2}\right)$$

for $1\le k\le n-2$, while

$$\mathbf{e}^1_{N-2}=\sqrt{\frac{n(1-q^2)}{A}}\sum_{b=2}^{n}\boldsymbol{\sigma}^b-\sqrt{\frac{n(1-q^2)}{A}}(n-1)q_1\boldsymbol{\sigma}^1 \\ -\sqrt{\frac{n}{A(1-q^2)}}(n-1)q(1-q_1)\left(q\boldsymbol{\sigma}^1-\boldsymbol{\sigma}^0\right)$$

with the proper normalization factor $A=n(n-1)(1-q_1)\left[1-nq^2+(n-1)q_1\right]$, and

$$\mathbf{e}^1_{N-1}=\frac{1}{\sqrt{1-q^2}}\left(q\boldsymbol{\sigma}^1-\boldsymbol{\sigma}^0\right).$$

This corresponds to choosing a unique vector, $\mathbf{e}^1_{N-1}$, having non-zero overlap with the fixed point $\boldsymbol{\sigma}^0$. Analogous choices can be made for any replica a with $a=2,\cdots,n$. As it will become clear in the following, this choice of bases is made to simplify the calculation of the conditional statistics of the Hessian.

C. Statistics of the conditioned Hessians (I)

In this section we discuss the statistics of the n Hessian matrices $\mathcal{H}^a$, conditioned to the gradients $\mathbf{g}^\alpha$ and to the energy fields h^α at the $n+1$ points $\boldsymbol{\sigma}^\alpha$. This is a necessary information to compute the joint expectation value in Eq. (4). We denote with $\tilde{\mathcal{H}}^a$ the matrices obeying this conditional law, and assume from now on that the overlaps between replicas satisfy the RS ansatz $q_{ab}\equiv q_1$.

As it follows from (9), the conditioned matrix $\tilde{\mathcal{H}}^a$ equals to

$$\tilde{\mathcal{H}}^a=\tilde{\mathcal{M}}^a-\sqrt{2N}p\epsilon\hat{1}, \tag{16}$$

where $\tilde{\mathcal{M}}^a$ is the Hessian projected onto the tangent plane, conditioned to gradients and energies. We aim at computing the covariances of the components $\tilde{\mathcal{M}}^a_{ij}$. We group all the independent components of the un-conditioned matrices $\mathcal{M}^a$ into an $nN(N+1)/2$-dimensional vector $\mathbf{M}=(\mathbf{M}_0,\mathbf{M}_{1/2},\mathbf{M}_1)$, where $\mathbf{M}_\gamma=(\mathbf{M}^1_\gamma,\cdots,\mathbf{M}^n_\gamma)$ for $\gamma\in\{0,1/2,1\}$. The vectors $\mathbf{M}_0,\mathbf{M}_1$ and $\mathbf{M}_{1/2}$ group the Hessians coordinates along directions that belong both to $S^\perp$, or both to S, or one to each subspace, respectively:

$$\mathbf{M}^a_0=(\mathcal{M}^a_{11},\mathcal{M}^a_{22},\cdots,\mathcal{M}^a_{MM},\mathcal{M}^a_{12},\cdots,\cdots,\mathcal{M}^a_{M-1M})$$
$$\mathbf{M}^a_{1/2}=(\mathcal{M}^a_{1M+1},\mathcal{M}^a_{1M+2},\cdots,\cdots,\cdots,\mathcal{M}^a_{MN-1})$$
$$\mathbf{M}^a_1=(\mathcal{M}^a_{M+1M+1},\cdots,\mathcal{M}^a_{N-1N-1},\mathcal{M}^a_{M+1M+2},\cdots),$$

where $M=N-n-1$. Analogously, we define the $(n+1)N$-dimensional vector $\tilde{\mathbf{g}}=(\tilde{\mathbf{g}}_0,\tilde{\mathbf{g}}_1)$, with $\tilde{\mathbf{g}}_\gamma=(\tilde{\mathbf{g}}^0_\gamma,\tilde{\mathbf{g}}^1_\gamma,\cdots,\tilde{\mathbf{g}}^n_\gamma)$, and:

$$\tilde{\mathbf{g}}^\alpha_0=(g^\alpha_1,\cdots,g^\alpha_M),$$
$$\tilde{\mathbf{g}}^\alpha_1=(g^\alpha_{M+1},\cdots,g^a_{N-1},\tilde{g}^\alpha_N).$$

Here $\tilde{g}^\alpha_N=\boldsymbol{\nabla}h^\alpha\cdot\boldsymbol{\sigma}^\alpha=p\,h^\alpha$, and thus conditioning to $h^a=\sqrt{2N}\epsilon$ and $h^0=\sqrt{2N}\epsilon_0$ is equivalent to conditioning to $\tilde{g}^a_N=\sqrt{2N}p\,\epsilon$ and $\tilde{g}^0_N=\sqrt{2N}p\,\epsilon_0$.

Before conditioning, the components in the block $\mathcal{M}^a_\gamma$ of the replica $\boldsymbol{\sigma}^a$ are correlated only with the component in the correspondent block $\mathcal{M}^b_\gamma$ of the other replicas $\boldsymbol{\sigma}^b$, since the covariance matrix $\mathbf{M}$ has a block-diagonal structure:

$$\hat{\Sigma}_{\mathbf{MM}}=\begin{pmatrix}\hat{\Sigma}^0_{\mathbf{MM}} & 0 & 0\\ 0 & \hat{\Sigma}^{1/2}_{\mathbf{MM}} & 0\\ 0 & 0 & \hat{\Sigma}^1_{\mathbf{MM}}\end{pmatrix}.$$

Since,

$$\hat{\Sigma}_{\tilde{\mathbf{g}}\tilde{\mathbf{g}}}=\begin{pmatrix}\hat{\Sigma}^0_{\tilde{\mathbf{g}}\tilde{\mathbf{g}}} & 0\\ 0 & \hat{\Sigma}^1_{\tilde{\mathbf{g}}\tilde{\mathbf{g}}}\end{pmatrix}$$

and the covariances between $\boldsymbol{M}$ and $\tilde{\mathbf{g}}$, see (13), are of the form:

$$\hat{\Sigma}_{\mathbf{M}\tilde{\mathbf{g}}}=\begin{pmatrix}0 & 0\\ \hat{\Sigma}^{\frac{1}{2}0}_{\mathbf{M}\tilde{\mathbf{g}}} & 0\\ 0 & \Sigma^{11}_{\mathbf{M}\tilde{\mathbf{g}}}\end{pmatrix},$$

this implies:

$$\hat{\Sigma}_{\mathbf{M}|\tilde{\mathbf{g}}}=\begin{pmatrix}\hat{\Sigma}^0_{\mathbf{MM}} & 0 & 0\\ 0\,\hat{\Sigma}^{1/2}_{\mathbf{MM}}-\hat{\Sigma}^{\frac{1}{2}0}_{\mathbf{M}\tilde{\mathbf{g}}}(\hat{\Sigma}^{-1}_{\tilde{\mathbf{g}}\tilde{\mathbf{g}}})^{00}\hat{\Sigma}^{0\frac{1}{2}}_{\tilde{\mathbf{g}}\mathbf{M}} & 0\\ 0 & 0 & \hat{\Sigma}^1_{\mathbf{MM}}-\hat{\Sigma}^{11}_{\mathbf{M}\tilde{\mathbf{g}}}(\hat{\Sigma}^{-1}_{\tilde{\mathbf{g}}\tilde{\mathbf{g}}})^{11}\hat{\Sigma}^{11}_{\tilde{\mathbf{g}}\mathbf{M}}\end{pmatrix}. \tag{17}$$

Thus the conditioning to the gradients and energies preserves this block-structure of the matrix elements; moreover, the covariances of the largest blocks $\mathcal{M}^a_0$ are left

untouched by the conditioning, since the subspace $S^{\perp}$ is blind to the presence of the other replicas. Thus, the components of this block form a GOE matrix with variance $\sigma^2 = p(p-1)$. This is the only relevant information to determine the expression of the action, see the following section. To characterize the stability of the stationary points counted by the complexity, instead, it is necessary to determine the conditional distribution of the remaining components.
For what concerns $\hat{\Sigma}^{1/2}_{\mathbf{M}|\tilde{\mathbf{g}}}$, we have:

$$\left(\hat{\Sigma}^{1/2}_{\mathbf{MM}}\right)^{ab}_{ij,kl} = \langle \mathcal{M}^a_{ij}\mathcal{M}^b_{kl}\rangle = \delta_{ik}S^{ab}_{jl},$$

where S^{ab} is a block of size $n\times n$, equal for every i, with components

$$\begin{aligned} S^{ab}_{jl} &= p(p-1)(p-2)q_1^{p-3}(\mathbf{e}^a_j\cdot\boldsymbol{\sigma}^b)(\mathbf{e}^b_l\cdot\boldsymbol{\sigma}^a) \\ &\quad + p(p-1)Q_{ab}^{p-2}(\mathbf{e}^a_j\cdot\mathbf{e}^b_l). \end{aligned} \tag{18}$$

Additionally, for $\beta = 0,\cdots,n$ it holds

$$\left(\hat{\Sigma}^{\frac{1}{2}0}_{\mathbf{M}\tilde{\mathbf{g}}}\right)^{a\beta}_{ij,k} = \langle \mathcal{M}^a_{ij}\, g^{\beta}_k\rangle = \delta_{ik}p(p-1)Q^{p-2}_{a\beta}(\mathbf{e}^a_j\cdot\boldsymbol{\sigma}^{\beta}), \tag{19}$$

and

$$(\hat{\Sigma}^0_{\tilde{\mathbf{g}}\tilde{\mathbf{g}}})^{-1} = \frac{1}{p}\begin{pmatrix} \alpha_0\hat{1} & \beta_0\hat{1} & \cdots & \cdots & \beta_0\hat{1} \\ \beta_0\hat{1} & \alpha_1\hat{1} & \beta_1\hat{1} & \cdots & \beta_1\hat{1} \\ \beta_0\hat{1} & \beta_1\hat{1} & \alpha_1\hat{1} & \cdots & \beta_1\hat{1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \beta_0\hat{1} & \beta_1\hat{1} & \cdots & \cdots & \alpha_1\hat{1} \end{pmatrix},$$

where the blocks have dimension $M\times M$, and

$$\begin{aligned} \alpha_0 &= 1 - \frac{nq^{2p-2}}{-1+nq^{2p-2}+q_1^{p-1}-nq_1^{p-1}} \\ \beta_0 &= \frac{q^{p-1}}{-1+nq^{2p-2}+q_1^{p-1}-nq_1^{p-1}} \\ \alpha_1 &= -\frac{1-(n-1)q^{2p-2}+(n-2)q_1^{p-1}}{(1-q_1^{p-1})(-1+nq^{2p-2}+q_1^{p-1}-nq_1^{p-1})} \\ \beta_1 &= \frac{q_1^{p-1}-q^{2p-2}}{(1-q_1^{p-1})(-1+nq^{2p-2}+q_1^{p-1}-nq_1^{p-1})}. \end{aligned} \tag{20}$$

Doing the matrix product, we find

$$\left(\hat{\Sigma}^{1/2}_{\mathbf{M}|\tilde{\mathbf{g}}}\right)^{ab}_{ik,jl} = \delta_{ij}\; p(p-1)T^{ab}_{kl}, \tag{21}$$

with

$$\begin{aligned} &T^{ab}_{kl} = Q^{p-2}_{ab}(\mathbf{e}^a_k\cdot\mathbf{e}^b_l) + (p-2)q_1^{p-3}(\mathbf{e}^a_k\cdot\boldsymbol{\sigma}^b)(\boldsymbol{\sigma}^a\cdot\mathbf{e}^b_l) - \\ &(p-1)\Big\{\alpha_0 q^{2p-4}(\mathbf{e}^a_k\cdot\boldsymbol{\sigma}^0)(\boldsymbol{\sigma}^0\cdot\mathbf{e}^b_l) + \\ &\beta_0(qq_1)^{p-2}\sum_{c=1}^{n}\left[(\mathbf{e}^a_k\cdot\boldsymbol{\sigma}^c)(\boldsymbol{\sigma}^0\cdot\mathbf{e}^b_l) + (\mathbf{e}^a_k\cdot\boldsymbol{\sigma}^0)(\boldsymbol{\sigma}^c\cdot\mathbf{e}^b_l)\right] + \\ &\frac{q_1^{2p-4}}{1-q_1^{p-1}}\sum_{c(\neq a,b)=1}^{n}(\mathbf{e}^a_k\cdot\boldsymbol{\sigma}^c)(\mathbf{e}^b_l\cdot\boldsymbol{\sigma}^c) + \\ &\beta_1 q_1^{2p-4}\sum_{c(\neq a)=1}^{n}\sum_{d(\neq b)=1}^{n}(\mathbf{e}^a_k\cdot\boldsymbol{\sigma}^c)(\mathbf{e}^b_l\cdot\boldsymbol{\sigma}^d)\Big\}, \end{aligned} \tag{22}$$

for $k,l = M+1,\cdots,N-1$. The averages of these components equals to zero after the conditioning, since they are proportional to the elements of $\tilde{\mathbf{g}}^{\alpha}_0$, which are all set to zero.
It remains to characterize the conditional distribution of the components $\mathbf{M}_1$. As it appears in the following, the covariances of these components do not enter in the stability analysis, while their non-zero averages induced by the conditioning do. We thus focus on the latter. Following the strategy illustrated in Ref.[1] and using the fact that, for each a, $\mathbf{e}^a_{N-1}$ is the only vector in the tangent plane at $\boldsymbol{\sigma}^a$ having non-zero overlap with $\boldsymbol{\sigma}^0$, we find:

$$\begin{aligned} \frac{\langle\tilde{M}^a_{ij}\rangle}{\sqrt{2N}} &= \lambda_1\delta_{i,j}\delta_{j,N-1} + \lambda_2\sum_{b(\neq a)}(\mathbf{e}^a_i\cdot\boldsymbol{\sigma}^b)(\mathbf{e}^a_j\cdot\boldsymbol{\sigma}^b) + \\ &\quad + \lambda_3\left(\delta_{i,N-1}\sum_{b(\neq a)}(\mathbf{e}^a_j\cdot\boldsymbol{\sigma}^b) + \delta_{j,N-1}\sum_{b(\neq a)}(\mathbf{e}^a_i\cdot\boldsymbol{\sigma}^b)\right) \\ &\quad + \lambda_4\sum_{b(\neq a)}(\mathbf{e}^a_i\cdot\boldsymbol{\sigma}^b)\sum_{c(\neq a)}(\mathbf{e}^a_j\cdot\boldsymbol{\sigma}^c), \end{aligned} \tag{23}$$

where λ_i are constants that depend explicitly on $q, q_1, \epsilon, \epsilon_0$ and n. Note that, with the choice of basis discussed in the previous section, it holds

$$\sum_{b\neq a}(\mathbf{e}^a_i\cdot\boldsymbol{\sigma}^b)(\mathbf{e}^a_j\cdot\boldsymbol{\sigma}^b) = \delta_{ij}(1-q_1) \tag{24}$$

and $\sum_{b\neq a}(\mathbf{e}^a_i\cdot\boldsymbol{\sigma}^b) = 0$ for any $i = M+1,\cdots,N-3$. The only non-zero averages to be determined are thus $\langle\tilde{M}^a_{ij}\rangle$ with $i,j\in\{N-2,N-1\}$. In the following, we will evaluate these average at the saddle point value for q_1, see Sec. F.

D. Derivation of the action

Having characterized the statistics of the conditioned Hessians, in this section we derive the explicit expression of $\Sigma(\epsilon,\hat{Q}|\epsilon_0)$ in (8).

The expression of the moments (4) can be rewritten as

$$M_n = \int \prod_{a<b=1}^{n} dQ_{ab}\, V_n(\hat{Q})\, \mathcal{E}_n(\epsilon, \hat{Q})\, P_n(\epsilon, \hat{Q}|\epsilon_0), \qquad (25)$$

where $\mathcal{E}_n$ and P_n are the expectation value and the joint distribution in Eq.(4), now expressed as a function of the $(n+1)\times(n+1)$ overlap matrix $\hat{Q}$, while V_n is an entropic contribution reading:

$$V_n(\hat{Q}) = \int \prod_{a=1}^{n} d\boldsymbol{\sigma}^a \delta\left(\boldsymbol{\sigma}^a \cdot \boldsymbol{\sigma}^0 - q\right) \prod_{a\leq b=1}^{n} \delta\left(Q_{ab} - \boldsymbol{\sigma}^a \cdot \boldsymbol{\sigma}^b\right). \qquad (26)$$

The explicit form of $\Sigma(\epsilon, \hat{Q}|\epsilon_0)$ is obtained extracting the leading order contribution in Nn of each of the three terms in (25). We consider each of them separately in the following three subsections, and collect all terms in the final expression, Eq.53. The calculation is done under the assumption that $q_{ab} \equiv q_1$.

1. Phase space term

The calculation of the phase-space term is standard, see also Ref. [1], and leads to:

$$V_n = e^{\frac{Nn}{2}\left[\log\left(\frac{2\pi e(1-q_1)}{N}\right) + \frac{q_1 - q^2}{1-q_1}\right] + o(Nn)}. \qquad (27)$$

Note that, for $q_1 = q^2$, the second term at the exponent vanishes, and the expression (27) reproduces the form of (26) for $n = 1$, which is the term obtained when performing the annealed calculation of the complexity.

2. Joint distribution of energies and gradients

The joint distribution of the gradients and energies of the n replicas $\boldsymbol{\sigma}^a$ conditioned to $\boldsymbol{\sigma}^0$ can be obtained as

$$P_n(\epsilon, \hat{Q}|\epsilon_0) = \frac{P_{n+1}(\epsilon, \epsilon_0, \hat{Q})}{P_1(\epsilon_0, \hat{Q})}, \qquad (28)$$

where $P_{n+1}(\epsilon, \epsilon_0, \hat{Q})$ is the joint distribution of the gradients and energies of the $n+1$ points $\boldsymbol{\sigma}^\alpha$, $\alpha = 0, 1, \cdots, n$, evaluated at $\mathbf{g}^\alpha = \mathbf{0}$, $h^a = \sqrt{2N}\epsilon$ and $h^0 = \sqrt{2N}\epsilon_0$, while $p_1(\epsilon_0, \hat{Q})$ is the density of the gradient and energy field of $\boldsymbol{\sigma}^0$, evaluated at $\mathbf{g}^0 = \mathbf{0}$ and $h^0 = \sqrt{2N}\epsilon_0$. From the fact that the gradient and energy field at the same point $\boldsymbol{\sigma}$ are uncorrelated, see (10), it follows that:

$$P_1(\epsilon_0, \hat{Q}) = \frac{e^{-N\epsilon_0^2}}{\sqrt{2\pi}} \frac{1}{(2\pi p)^{(N-1)/2}} = e^{-N\left(\epsilon_0^2 + \frac{\log(2\pi p)}{2}\right) + o(N)}. \qquad (29)$$

To compute $P_{n+1}(\epsilon, \epsilon_0, \hat{Q})$, it is convenient to proceed as in Ref. [1] and first determine the joint distribution of the N-dimensional vectors $\tilde{\mathbf{g}}[\boldsymbol{\sigma}^\alpha] \equiv \tilde{\mathbf{g}}^\alpha = (\tilde{g}_0^\alpha, g_1^\alpha, g_2^\alpha, \cdots, g_{N-1}^\alpha)$, whose last $N-1$ components are the components of the gradient $\boldsymbol{\nabla} h^\alpha$ in the chosen basis $\mathcal{B}[\boldsymbol{\sigma}^\alpha]$ of the tangent plane at $\boldsymbol{\sigma}^\alpha$, $g_\beta^\alpha = \boldsymbol{\nabla} h[\boldsymbol{\sigma}^\alpha] \cdot \mathbf{e}_\beta^\alpha$, while the first component is proportional to the energy field, $\tilde{g}_0^\alpha = \boldsymbol{\nabla} h[\boldsymbol{\sigma}^\alpha] \cdot \boldsymbol{\sigma}^\alpha = p\, h[\boldsymbol{\sigma}^\alpha]$. The joint density of the vectors $\tilde{\mathbf{g}}^\alpha$ evaluated at $\tilde{g}_i^\alpha = 0$ for $i = 1, \cdots, N-1$ equals to:

$$P\left(\{\tilde{g}_0^\alpha, \mathbf{0}\}_{\alpha=0}^n\right) = \frac{\exp\left\{-\frac{1}{2}\sum_{\alpha,\beta=0}^n \tilde{g}_0^\alpha \boldsymbol{\sigma}^\alpha \cdot [\hat{C}^{-1}]^{\alpha\beta} \cdot \boldsymbol{\sigma}^\beta \tilde{g}_0^\beta\right\}}{(2\pi)^{\frac{(n+1)N}{2}} |\det \hat{C}|^{\frac{1}{2}}}, \qquad (30)$$

where $\hat{C}$ is the covariance matrix of the gradients in the reference frame of the extended N-dimensional space,

$$C_{ij}^{\alpha\beta} \equiv \langle \boldsymbol{\nabla} h_i^\alpha \boldsymbol{\nabla} h_j^\beta \rangle = p Q_{\alpha\beta}^{p-1} \delta_{ij} + p(p-1) Q_{\alpha\beta}^{p-2} \sigma_j^\alpha \sigma_i^\beta. \qquad (31)$$

Performing the change of variables at the exponent, we obtain

$$P_{n+1}(\epsilon, \epsilon_0, \hat{Q}) = \frac{p^{2n+2}}{(2\pi)^{\frac{(n+1)N}{2}} |\det \hat{C}|^{\frac{1}{2}}} e^{-Np^2 f(\epsilon, \epsilon_0, q_1, q)}, \qquad (32)$$

where

$$f(\epsilon, \epsilon_0, q_1, q) = \epsilon^2 \sum_{a,b=1}^{n} M^{ab} + \epsilon_0^2 M^{00} + \epsilon_0 \epsilon \sum_{a=1}^{n} \left(M^{a0} + M^{0a}\right) \qquad (33)$$

and

$$M^{\alpha\beta} \equiv (\boldsymbol{\sigma}^\alpha)^T \cdot [\hat{C}^{-1}]^{\alpha\beta} \cdot \boldsymbol{\sigma}^\beta. \qquad (34)$$

The contribution of the determinant in (32) is easily obtained from the fact that $\hat{C}^{\alpha\beta} = \mathrm{diag}(\hat{A}^{\alpha\beta}, \hat{B}^{\alpha\beta})$, where $\hat{A}^{\alpha\beta}$ is the $(N-n-1)\times(N-n-1)$ block which gives the covariances between the gradients components in $S^\perp$, $\hat{A}_{ij}^{\alpha\beta} = p\delta_{ij}\left\{\delta_{\alpha\beta} + (1-\delta_{\alpha\beta})[q_1^{p-1} + (\delta_{\alpha 0} + \delta_{\beta 0})(q^{p-1} - q_1^{p-1})]\right\}$, while $\hat{B}^{\alpha\beta}$ are $(n+1)\times(n+1)$ blocks whose elements are the covariances of the gradients components in S. To leading order in N only the block $\hat{A}$ contributes, giving:

$$|\det \hat{C}| = p^N\, e^{Nn\left(\log\left[p(1-q_1^{p-1})\right] + \frac{q_1^{p-1} - q^{2p-2}}{1-q_1^{p-1}}\right) + o(Nn)}. \qquad (35)$$

To compute the quadratic form (33), it is convenient to introduce the set of $N(n+1)$-dimensional vectors:

$$\begin{aligned}
\vec{\boldsymbol{\xi}}_1 &= \left(\boldsymbol{\sigma}^0, \mathbf{0}, \cdots, \mathbf{0}\right), \\
\vec{\boldsymbol{\xi}}_2 &= \left(\sum_{a=1}^{n} \boldsymbol{\sigma}^a, \mathbf{0}, \cdots, \mathbf{0}\right), \\
\vec{\boldsymbol{\xi}}_3 &= \left(\mathbf{0}, \boldsymbol{\sigma}^1, \cdots, \boldsymbol{\sigma}^n\right), \\
\vec{\boldsymbol{\xi}}_4 &= \left(\mathbf{0}, \boldsymbol{\sigma}^0, \cdots, \boldsymbol{\sigma}^0\right), \\
\vec{\boldsymbol{\xi}}_5 &= \left(\mathbf{0}, \sum_{a\neq\{0,1\}} \boldsymbol{\sigma}^a, \cdots, \sum_{a\neq\{0,n\}} \boldsymbol{\sigma}^a\right),
\end{aligned} \qquad (36)$$

which form a close set under the action of the matrix $\hat{C}^{-1}$. To show that this is the case, we split the covariance matrix into its diagonal $\hat{D}$ and off-diagonal $\hat{O}$ parts in the space of replicas, $\hat{C} = p\left(\hat{O} + \hat{D}\right)$, and write:

$$\hat{C}^{-1} = p^{-1}\hat{D}^{-1}\left(\hat{1} + \hat{O}\hat{D}^{-1}\right)^{-1} \tag{37}$$

where

$$[\hat{D}^{-1}]_{ij}^{\alpha\beta} = \delta_{\alpha\beta}\left(\delta_{ij} - \frac{p-1}{p}\sigma_i^\alpha\sigma_j^\alpha\right), \tag{38}$$

and

$$\begin{aligned}
&[\hat{O}\hat{D}^{-1}]_{ij}^{\alpha\beta} = \\
&(1-\delta_{\alpha\beta})(\delta_{\alpha 0}+\delta_{\beta 0})\left[A'\delta_{ij} + B'\sigma_i^\beta\sigma_j^\alpha - C'\sigma_i^\beta\sigma_j^\beta\right] + \\
&(1-\delta_{\alpha\beta})(1-\delta_{\alpha 0}-\delta_{\beta 0})\left[A\delta_{ij} + B\sigma_i^\beta\sigma_j^\alpha - C\sigma_i^\beta\sigma_j^\beta\right],
\end{aligned}$$

with $A' = q^{p-1}, B' = (p-1)q^{p-2}, C' = (p-1)q^{p-1}$ and $A = q_1^{p-1}, B = (p-1)q_1^{p-2}, C = (p-1)q_1^{p-1}$. Then it is immediate to show that:

$$\begin{aligned}
\hat{O}\hat{D}^{-1}\vec{\boldsymbol{\xi}}_1 &= A'\vec{\boldsymbol{\xi}}_4, \\
\hat{O}\hat{D}^{-1}\vec{\boldsymbol{\xi}}_2 &= A'\vec{\boldsymbol{\xi}}_3 + \left[B' - qC' + (n-1)(B'q_1 - C'q)\right]\vec{\boldsymbol{\xi}}_4 \\
&\quad + A'\vec{\boldsymbol{\xi}}_5, \\
\hat{O}\hat{D}^{-1}\vec{\boldsymbol{\xi}}_3 &= A'\vec{\boldsymbol{\xi}}_2 + A\vec{\boldsymbol{\xi}}_5, \\
\hat{O}\hat{D}^{-1}\vec{\boldsymbol{\xi}}_4 &= A'\vec{\boldsymbol{\xi}}_1 + (B' - C'q)\vec{\boldsymbol{\xi}}_2 + A(n-1)\vec{\boldsymbol{\xi}}_4 \\
&\quad + (B-C)q\vec{\boldsymbol{\xi}}_5, \\
\hat{O}\hat{D}^{-1}\vec{\boldsymbol{\xi}}_5 &= (n-1)\left[A' + B'q - C'q_1\right]\vec{\boldsymbol{\xi}}_2 + A(n-1)\vec{\boldsymbol{\xi}}_3 \\
&\quad + \left[A(n-2) + B - Cq_1 + (n-2)q_1(B-C)\right]\vec{\boldsymbol{\xi}}_5.
\end{aligned}$$

To compute the action of $\left(\hat{1} + \hat{O}\hat{D}^{-1}\right)^{-1}$ in this closed subspace, we introduce an orthonormal basis for it given by the vectors:

$$\begin{aligned}
\vec{\boldsymbol{\chi}}_1 &= \vec{\boldsymbol{\xi}}_1 \\
\vec{\boldsymbol{\chi}}_2 &= \frac{1}{\sqrt{n(1-nq^2+(n-1)q_1)}}\left(-nq\vec{\boldsymbol{\xi}}_1 + \vec{\boldsymbol{\xi}}_2\right) \\
\vec{\boldsymbol{\chi}}_3 &= \frac{1}{\sqrt{n}}\vec{\boldsymbol{\xi}}_3 \\
\vec{\boldsymbol{\chi}}_4 &= \frac{1}{\sqrt{n(1-q^2)}}\left(-q\vec{\boldsymbol{\xi}}_3 + \vec{\boldsymbol{\xi}}_4\right) \\
\vec{\boldsymbol{\chi}}_5 &= \frac{1}{\sqrt{n(n-1)(1-q^2)(1-q_1)(1-nq^2+(n-1)q_1)}}\times \\
&\times\left((n-1)(q^2-q_1)\vec{\boldsymbol{\xi}}_3 - (n-1)q(1-q_1)\vec{\boldsymbol{\xi}}_4 + (1-q^2)\vec{\boldsymbol{\xi}}_5\right).
\end{aligned} \tag{39}$$

In this basis, the action of the operator $\hat{1}+\hat{O}\hat{D}^{-1}$ is given by the following matrix:

$$\hat{1} + \hat{O}\hat{D}^{-1} = \left[\begin{array}{c|cc} \overset{2\times 2}{\hat{1}} & \multicolumn{2}{c}{\overset{2\times 3}{L_1}} \\ \hline \overset{3\times 2}{L_2} & \multicolumn{2}{c}{\overset{3\times 3}{\hat{1}} + \overset{3\times 3}{L_3}} \end{array}\right]$$

with blocks

$$L_1 = \begin{pmatrix} \sqrt{n}q^p & pq^{p-1}\sqrt{n-nq^2} & 0 \\ q^{p-1}S_1 & \frac{S_1 q^{p-2}\left[p(1-q^2)-1\right]}{\sqrt{1-q^2}} & q^{p-1}\sqrt{\frac{(n-1)(1-q_1)}{1-q^2}} \end{pmatrix},$$

$$L_2 = \begin{pmatrix} \sqrt{n}q^p & pq^{p-1}S_1 \\ q^{p-1}\sqrt{n-nq^2} & \frac{S_1 q^{p-2}\left[p(1-q^2)-1\right]}{\sqrt{1-q^2}} \\ 0 & q^{p-1}\sqrt{\frac{(n-1)(1-q_1)}{1-q^2}} \end{pmatrix},$$

and

$$L_3 = \begin{pmatrix} (n-1)q_1^p & p\frac{(n-1)q(1-q_1)q_1^{p-1}}{\sqrt{1-q^2}} & \frac{S_2 p q_1^{p-1}}{\sqrt{1-q^2}} \\ \frac{(n-1)q(1-q_1)q_1^{p-1}}{\sqrt{1-q^2}} & S_3 & \frac{q q_1^{p-2}[p(1-q_1)-1]S_2}{\sqrt{1-q^2}} \\ \frac{S_2 q_1^{p-1}}{\sqrt{1-q^2}} & \frac{S_2 q q_1^{p-2}[p(1-q_1)-1]}{\sqrt{1-q^2}} & \frac{q_1^{p-2}S_4}{-1+q^2}. \end{pmatrix}$$

where

$$\begin{aligned}
S_1 &= \sqrt{1-nq^2+(n-1)q_1}, \\
S_2 &= \sqrt{(n-1)(1-q_1)(1-nq^2+(n-1)q_1)}, \\
S_3 &= \frac{(n-1)q_1^{p-2}(q^2(p(q_1-1)^2-1)+q_1)}{1-q^2}, \\
S_4 &= 1 - nq^2 + (n-1)q^2q_1 - p(1-q_1)(1-q_1+n(q_1-q^2)).
\end{aligned}$$

Setting

$$\hat{Y} \equiv \left(\hat{1} + \hat{O}\hat{D}^{-1}\right)^{-1} \tag{40}$$

for the inverse of this matrix, we get that the quadratic form in (33) can be written in terms of its matrix elements in the basis (39), as

$$p^2 f(\epsilon, \epsilon_0, q_1, q) = \epsilon_0^2 Y_{11} + \epsilon_0\epsilon\sqrt{n}\left[Y_{13} + Y_{31}\right] + \epsilon^2 n Y_{33}, \tag{41}$$

where the Y_{ij} depend on q, q_1 and n.
The expression for the Y_{ij} for general n is rather cumbersome. A major simplification occurs for $n \to 1$, where only one replica is present. In this case the dependence on the overlap q_1 naturally drops, and one gets:

$$\begin{aligned}
Y_{11} &\to \frac{q^4 - q^{2p}[1 + p(p-2+(3-2p)q^2 + (p-1)q^4)]}{Y} \\
Y_{13} + Y_{31} &\to \frac{-2q^{p+4} + 2q^{3p}(1-p(1-q^2))}{Y} \\
Y_{33} &\to \frac{q^4 - q^{2p}(1 + p(p-2+(3-2p)q^2 + (p-1)q^4))}{Y}
\end{aligned} \tag{42}$$

with the denominator being equal to

$$Y = q^4 + q^{4p} - q^{2p}[(p-1)^2 - 2(p-2)pq^2 + (p-1)^2 q^4]. \quad (43)$$

The joint density of gradients and energies of the two stationary points $\boldsymbol{\sigma}^0$ and $\boldsymbol{\sigma}^1$ is in this case is equal to

$$P_2(\epsilon, \epsilon_0, q) = \left(\frac{e^{-\left(\epsilon_0^2 Y_{11} + \epsilon_0 \epsilon (Y_{13}+Y_{31}) + \epsilon^2 Y_{33}\right)}}{2\pi p \sqrt{1-q^{2p-2}}} \right)^N e^{o(N)}. \quad (44)$$

This is the contribution that one gets from the annealed calculation of the complexity, which, as we show in the following, is reproduced by the quenched calculation evaluated at the saddle point for q_1.

To compute the contribution to the quenched complexity, we consider the expansion of the matrix elements of $\hat{Y}$ to linear order in n:

$$\begin{aligned} Y_{11} &= 1 + n\frac{y_{11}(q_1,q)}{y(q_1,q)} + o(n), \\ \sqrt{n}\,[Y_{13}+Y_{31}] &= n\frac{y_{13}(q_1,q)}{y(q_1,q)} + o(n), \\ nY_{33} &= n\frac{y_{33}(q_1,q)}{y(q_1,q)} + o(n) \end{aligned} \quad (45)$$

with

$$\begin{aligned} y_{11} &= -p^2 q^{2p+2}(1-q_1)^2 q_1^{p+1} \\ &\quad + q^{2p} q_1^3 (1-q_1^{p-2}) \left(q^4 (q_1^{p-1}-1) + (1-q_1) q^{2p} \right) \\ &\quad - pq^{2p} q_1^3 \left(1 - q_1^{p-2}\right) \times \\ &\quad \times \left[(1-q_1) q^{2p} - q^4 \left(1 - q_1^{p-1}\right) + q^2 \left(1 - q_1^p\right) \right], \\ y_{13} &= 2q^p [-q_1^2 + q_1^p (1 - p(1-q_1))] \times \\ &\quad \times [-(p-1) q^{2p} (1-q_1) q_1 + q^4 q_1 (q_1^{p-1} - 1)], \\ y_{33} &= p^2 q^{2+2p} (1-q_1) q_1^3 (q_1^{p-1} - 1) \\ &\quad + (p-1)^2 q^{2p} (1-q_1) q_1 [q_1^2 + q_1^p (p(q_1-1)^2 - 1)] \\ &\quad + q^4 q_1 (q_1^{p-1} - 1)[q_1^2 + q_1^p (p(1-q_1)^2 - 1)], \\ y &= -p(p-1) q^{2p+2} (1-q_1) q_1^3 (1 - q_1^{p-1})(1 - q_1^{p-2}) \\ &\quad + (p-1)^2 (1-q_1) q_1 q^{2p} \times \\ &\quad \times \left(q_1^{2p} + \left(p(1-q_1)^2 - q_1^2 - 1 \right) q_1^p + q_1^2 \right) + \\ &\quad q^4 \left(q_1^p - q_1\right) \left(q_1^{2p} + \left(p(1-q_1)^2 - q_1^2 - 1 \right) q_1^p + q_1^2 \right). \end{aligned} \quad (46)$$

Combining (35) and (41) we get:

$$P_n(\epsilon, \hat{Q}|\epsilon_0) = \frac{e^{-\frac{Nn}{2}\left[F(\epsilon,\epsilon_0) + \frac{q_1^{p-1} - q^{2p-2}}{1-q_1^{p-1}}\right] + o(Nn)}}{\left[2\pi p (1-q_1^{p-1})\right]^{\frac{Nn}{2}}}, \quad (47)$$

where the linearized quadratic form is given by:

$$F(\epsilon, \epsilon_0) = \frac{2}{y}\left[\epsilon_0^2 y_{11} + \epsilon_0 \epsilon y_{13} + \epsilon^2 y_{33}\right]. \quad (48)$$

3. Expectation value of the determinants

The expectation value $\mathcal{E}_n(\epsilon, \hat{Q})$ is over the joint distribution of the Hessians of the n replicas $\boldsymbol{\sigma}^a$, conditioned to the values of the gradients and energy fields of all the $n+1$ points $\boldsymbol{\sigma}^\alpha$. Following exactly the same steps as in Ref. [1], we can argue that:

(i) even though the conditioned Hessian matrices $\tilde{\mathcal{H}}^a$ associated to different replicas are correlated with each others, these correlations are irrelevant when computing the leading-order term in N of $\mathcal{E}_n(\epsilon, \hat{Q})$, as it holds:

$$\mathcal{E}_n = \left\langle \prod_{a=1}^n |\det \tilde{\mathcal{H}}^a| \right\rangle = e^{\sum_{a=1}^n \langle |\det \tilde{\mathcal{H}}^a| \rangle + o(N)}. \quad (49)$$

The reason for this equality (valid at leading exponential order in N) is that the joint probability measure on the eigenvalue densities has the form of a large deviation principle in e^{N^2}. In consequence, the average above does not bias the measure at leading exponential order in N and one can replace the average of the exponential with the exponential of the average. See Ref. [1] for a detailed explanation. Given the equivalence between replicas, Eq. (49) can be written as:

$$\mathcal{E}_n = N^{\frac{Nn}{2}} e^{Nn \int d\lambda \rho_{\rm sp}(\lambda) \log|\lambda| + o(Nn)}, \quad (50)$$

where $\rho_{\rm sp}(\lambda)$ is the density of states of the matrices $\tilde{\mathcal{H}}^a/\sqrt{N}$.

(ii) The exponent (50) is, to leading order in N, determined by the bulk of the density of states $\rho_{\rm sp}(\lambda)$. This is governed by the largest $(N-n-1)\times(N-n-1)$ block of the Hessian, whose components are iid Gaussian variables with variance $\sigma^2 = p(p-1)$ and non-zero average along the diagonal, due to the shift in (16). As a result, up to subleading corrections in $1/N$ it holds:

$$\rho_{\rm sp}(\lambda) = \frac{\sqrt{4p(p-1) - (\lambda + \sqrt{2}p\epsilon)^2}}{2\pi p(p-1)}. \quad (51)$$

Combining these two results, we obtain

$$\mathcal{E}_n(\epsilon, \hat{Q}) = e^{\frac{Nn}{2}\left[\log N + \log[2p(p-1)] + 2I\left(\sqrt{\frac{p}{p-1}}\epsilon\right)\right] + o(Nn)}, \quad (52)$$

where $I(y) = I(-y)$ is given by:

$$I = \begin{cases} \frac{y^2-1}{2} + \frac{y}{2}\sqrt{y^2-2} + \log\left(\frac{-y+\sqrt{y^2-2}}{2}\right) \text{ if } y \le -\sqrt{2}, \\ \frac{1}{2}y^2 - \frac{1}{2}(1+\log 2) \text{ if } -\sqrt{2} \le y \le 0. \end{cases}$$

Because of the factorization in (49), this contribution is independent on the overlap q_1 between replicas, and it is equal to the contribution one would get from the annealed calculation, elevated to the power n.

E. Saddle point of the action and equivalence to annealed

Combining the results (27), (47) and (52), we get that the linear order term in (8) reads:

$$\Sigma(\epsilon, q, q_1|\epsilon_0) = \frac{1}{2}\begin{cases}\Sigma_<(\epsilon, q, q_1|\epsilon_0) & \text{if } \epsilon \leq \epsilon_{\rm th}(p)\\ \Sigma_>(\epsilon, q, q_1|\epsilon_0) & \text{if } \epsilon > \epsilon_{\rm th}(p)\end{cases} \tag{53}$$

where $\epsilon_{\rm th}(p) = -\sqrt{2(p-1)/p}$ and

$$\begin{aligned}\Sigma_< &= \log\left(\frac{p}{2}\right) + \frac{p}{p-1}\left(\epsilon^2 + \epsilon\tilde{z}\right) + 2\log\left(-\epsilon + \tilde{z}\right) + Q,\\ \Sigma_> &= \log(p-1) + \frac{p}{p-1}\epsilon^2 + Q,\end{aligned} \tag{54}$$

with $\tilde{z} = \sqrt{\epsilon^2 - \epsilon_{\rm th}^2}$ and where Q is the only term depending explicitly on q_1,

$$Q = \log\left(\frac{1-q_1}{1-q_1^{p-1}}\right) + \frac{q_1 - q^2}{1-q_1} + \frac{q^{2p-2} - q_1^{p-1}}{1-q_1^{p-1}} - F(\epsilon, \epsilon_0). \tag{55}$$

The saddle point value for q_1 is therefore determined by the equation $\partial Q/\partial q_1 = 0$; if multiple solutions are present, the global minimum should be selected. We find that, irrespectively of the values of ϵ and ϵ_0, $q_1 = q^2$ is always a solution to this equation. For q sufficiently large, a second minimum appears, which for certain ϵ is the deepest one, see Fig. 1. However, when this happens the corresponding complexity is found to be always smaller than zero, corresponding to the absence of stationary points. In conclusion, we find that the relevant

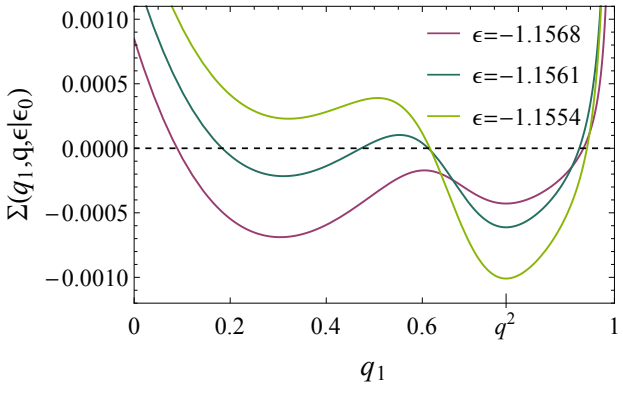


FIG. 1. Complexity as a function of the overlap q_1 for $q = 0.88$ and $\epsilon_0 = -1.1582$ and different values of ϵ.

saddle point solutions for q_1 is $q_1 = q^2$. This has a simple geometrical interpretation: q^2 is the minimal possible overlap between vectors on the sphere that are constrained to be at fixed overlap q with a fixed direction; it corresponds to the n replicas having zero overlap with each others in the subspace orthogonal to the direction singled out by $\boldsymbol{\sigma}^0$. When plugging this value into (53), we find that the quenched complexity reproduces the annealed one, which is obtained taking the logarithm of the average number of stationary points at fixed overlap with a minimum $\boldsymbol{\sigma}^0$. In particular, (55) reduces to:

$$Q \to \log\left(\frac{1-q^2}{1-q^{2p-2}}\right) - \left(\epsilon_0^2 U_0(q) + \epsilon_0\epsilon U(q) + \epsilon^2 U_1(q)\right), \tag{56}$$

with

$$\begin{aligned}U_0(q) &= \frac{q^{2p}\left(-q^{2p} + p\left(q^2 - q^4\right) + q^4\right)}{q^{4p} - \left((p-1)^2(1+q^4) - 2(p-2)pq^2\right)q^{2p} + q^4},\\ U(q) &= \frac{2q^{3p}\left(p\left(q^2-1\right)+1\right) - 2q^{p+4}}{q^{4p} - \left((p-1)^2(1+q^4) - 2(p-2)pq^2\right)q^{2p} + q^4},\\ U_1(q) &= \frac{q^4 - q^{2p}\left(p\left((p-1)q^4 + (3-2p)q^2 + p - 2\right) + 1\right)}{q^{4p} - \left((p-1)^2(1+q^4) - 2(p-2)pq^2\right)q^{2p} + q^4}.\end{aligned} \tag{57}$$

These expressions reproduce the limit of (41) when $n \to 1$, and thus are equally obtained when performing the annealed calculation for the complexity (see e.g. [6]).

F. Statistics of the conditioned Hessian (II): at the saddle point

1. *Variances and averages at the saddle point*

We now discuss the statistics of the Hessian matrices, evaluated at the saddle point value for q_1. Setting $q_1 = q^2$, we find from (20) that $\beta_1 = 0$, and that (22) for $a = b$ reduces to

$$T_{kl}^{aa} = \delta_{kl}\left\{1 - c(q)\left[\delta_{kN-1} + (1-\delta_{kN-1})q^{2p-4}\right]\right\} \tag{58}$$

with

$$c(q) = \frac{(p-1)(1-q^2)q^{2p-4}}{1-q^{2p-2}}. \tag{59}$$

For what concerns the averages in (23), we find instead that for $q_1 = q^2$, the constant λ_4 vanishes, while $\sum_{b(\neq a)}(\mathbf{e}_{N-2}^a \cdot \sigma^b)^2 = (1-q^2)$. Therefore, in this limit the average $\langle \tilde{M}_{N-2\,N-2}^a\rangle/\sqrt{N}$ becomes equal to the ones of the components $\langle \tilde{M}_{ii}^a\rangle/\sqrt{N}$ for $i = M+1, \cdots, N-3$, see (24), and it is given by:

$$\nu \equiv \sqrt{2}\kappa_2(q, \epsilon, \epsilon_0)(1-q^2), \tag{60}$$

where $\kappa_2(q, \epsilon, \epsilon_0) = \lim_{q_1 \to q^2}\lambda_2(q, q_1, \epsilon, \epsilon_0, n)$ is independent on n and reads explicitly:

$$\kappa_2(q, \epsilon, \epsilon_0) = \frac{p(p-1)\kappa_2^{(n)}(q, q_1, \epsilon, \epsilon_0)}{\kappa_2^{(d)}(q, q_1, \epsilon, \epsilon_0)} \tag{61}$$

and

$$\kappa_2^{(d)} = q^4\left(q^{4-2p} + q^{2p} - (p-1)^2(1+q^4) + 2(p-2)pq^2\right)$$
$$\kappa_2^{(n)} = \epsilon\left[\left((p-2)q^2 - p + 1\right)q^{2p} + q^4\right] + \epsilon_0\left[\left((p-1)q^2 - p + 2\right)q^{p+2} - q^{3p}\right].$$

Moreover, for $q_1 = q^2$ we find $\lambda_3 - q\sqrt{1-q^2}\lambda_2 = 0$, implying that $\langle \tilde{M}^a_{N-1\,N-2}\rangle = 0$. The remaining non-zero average to be computed is $\langle \tilde{M}^a_{N-1\,N-1}\rangle/\sqrt{N} = \sqrt{2}[\lambda_1 + (n-1)q\sqrt{1-q^2}\left(q\sqrt{1-q^2}\lambda_2 - 2\lambda_3\right)]$, which at the saddle point is n-independent and explicitly equals to:

$$\mu(q,\epsilon,\epsilon_0) \equiv \frac{\sqrt{2}(p-1)p\left(1-q^2\right)\left(a_0(q)\epsilon_0 - a_1(q)\epsilon\right)}{a_2(q)} \tag{62}$$

with

$$a_1 = q^{3p} + q^{p+2}\left(p - 2 - (p-1)q^2\right)$$
$$a_0 = q^4 + q^{2p}\left(1 - p + (p-2)q^2\right)$$
$$a_2 = q^{6-p} + q^{3p+2} - q^{p+2}\left((p-1)^2(q^4+1) - 2(p-2)pq^2\right). \tag{63}$$

2. Structure of the Hessian at the saddle point

It follows from this that, at the saddle point $q_1 = q^2$, for each replica a the shifted Hessian can be written as:

$$\frac{\tilde{\mathcal{M}}}{\sqrt{N}} = \frac{\mathcal{S}}{\sqrt{N}} + \mathcal{D}, \tag{64}$$

where $\mathcal{S}$ is a stochastic matrix with the block structure:

$$\mathcal{S} = \begin{pmatrix} \mathcal{S}_0 & \mathcal{S}_{1/2} \\ \mathcal{S}_{1/2}^T & \mathcal{S}_1 \end{pmatrix}, \tag{65}$$

where the largest $(N-1-n)\times(N-1-n)$ block $\mathcal{S}_0$ is a GOE with $\sigma^2 = p(p-1)$, $\mathcal{S}_{1/2}$ is an $(N-n-1)\times n$ block with iid Gaussian entries

$$\mathcal{S}_{1/2} = \begin{pmatrix} n_{1M+1}\cdots n'_{1N-1} \\ n_{2M+1}\cdots n'_{2N-1} \\ \cdots\quad\cdots\quad\cdots \\ \cdots\quad\cdots\quad\cdots \\ n_{MM+1}\cdots n'_{MN-1} \end{pmatrix}, \tag{66}$$

where the n_{ij} have variance $\delta^2(q) = p(p-1)[1 - c(q)q^{2p-4}]$, while the n'_{iN-1} have yet another variance:

$$\Delta^2(q) = p(p-1)[1-c(q)]. \tag{67}$$

The conditioning thus reduces the fluctuations of these matrix elements with respect to the unconditioned case. The smaller $n\times n$ block $\mathcal{S}_0$ has entries that are mutually correlated, with non-zero averaged contained in the deterministic matrix $\mathcal{D}$. The latter has also a block structure $\mathcal{D} = \mathrm{diag}\,(\mathcal{D}_0, \mathcal{D}_1)$, with $\mathcal{D}_0 = 0$ and

$$\mathcal{D}_1 = \begin{pmatrix} \nu & 0 & \cdots & \cdots & 0 \\ 0 & \nu & \cdots & \cdots & 0 \\ 0 & 0 & \nu & \cdots & 0 \\ 0 & \cdots & \cdots & \nu & 0 \\ 0 & \cdots & \cdots & 0 & \mu \end{pmatrix} \tag{68}$$

with ν, μ given in (60), (62). The conditional Hessian $\tilde{H}/\sqrt{N}$ is obtained after a shift with a diagonal matrix, see Eq. (16). It is thus a shifted Gaussian matrix, perturbed with finite rank perturbation. Notice that in the annealed case (*i.e.*, for $n\to 1$), only one special line and column remain (the last one). As we shall now see, from the point of view of the isolated eigenvalue these are indeed the only column and row that matter; thus, even at the level of the eigenvalue the quenched calculation reproduces the annealed one.

3. Computation of the isolated eigenvalue

In the large-N limit, the bulk of the density of eigenvalues of $\tilde{\mathcal{M}}/\sqrt{N}$ is controlled by the GOE block, and is thus a centered semicircle. To discuss the stability of the stationary points, we need to compute the lower order corrections to this density of states, to determine whether there are isolated eigenvalues that become negative, inducing an instability. To this aim, we need to determine the poles of the resolvent of $\tilde{\mathcal{M}}/\sqrt{N}$ that lie on the real axis and are smaller than $-2\sqrt{p(p-1)}$. This requires to compute the trace of $(z - \tilde{\mathcal{M}}/\sqrt{N})^{-1}$. We focus on the contribution to the trace coming from the small $n\times n$ block of the resolvent; indeed, the corresponding matrix elements are the ones having non-zero overlap with the fixed minimum $\boldsymbol{\sigma}^0$, and thus only the poles of this part of the resolvent can be generated by the conditioning and can have eigenvectors with a non-zero component in the direction of the fixed minimum. The quantity to determine are therefore the poles of $\langle \mathrm{Tr}\,\{1/N\cdot D(z)\}\rangle$, where

$$D(z) \equiv z\hat{1} - \frac{\mathcal{S}_1}{\sqrt{N}} - \frac{1}{N}\mathcal{S}_{1/2}^T\left(z\hat{1} - \frac{\mathcal{S}_0}{\sqrt{N}}\right)^{-1}\mathcal{S}_{1/2}, \tag{69}$$

and where now the average is over the distribution of the entries of the matrix $\mathcal{S}$. Following the same step as in Ref. [1], we find that the poles are solutions of the equation:

$$\left[z - \nu - \delta^2 G_\sigma(z)\right]^{n-1}\left(z - \mu - \Delta^2 G_\sigma(z)\right) = 0, \tag{70}$$

where

$$G_\sigma(z) = \frac{z + \sqrt{z^2 - 4\sigma^2}}{2\sigma^2} \tag{71}$$

is the resolvent of a GOE matrix with variance σ^2. In particular, for $n\to 0$ we can focus on the solutions of

$z-\mu-\Delta^2 G_\sigma(z)=0$, which satisfy:

$$z\left(1-\frac{1}{2}\frac{\Delta^2(q)}{\sigma^2}\right)-\mu(q,\epsilon,\epsilon_0)=\frac{1}{2}\frac{\Delta^2(q)}{\sigma^2}\sqrt{z^2-4\sigma^2}. \tag{72}$$

We notice that for fixed q and ϵ_0, $\mu<0$ is a decreasing function of ϵ: this already indicates that the additive part of the rank-1 perturbation is stronger for stationary points that are at higher energy, that are therefore more prone to an instability towards $\boldsymbol{\sigma}^0$.
Taking the square of (72), we obtain a second order equation for z,

$$z^2\left(1-\frac{\Delta^2}{\sigma^2}\right)-2\mu\left(1-\frac{\Delta^2}{2\sigma^2}\right)z+\left(\mu^2+\frac{\Delta^4}{\sigma^2}\right)=0. \tag{73}$$

Of the two solutions $z_\pm(q,\epsilon,\epsilon_0)$ of this equations (differing for the sign in front of the square root), only those that are real and satisfy

$$z_\pm\left(1-\frac{\Delta^2}{2\sigma^2}\right)-\mu\geq 0 \tag{74}$$

have to be retained, as they are consistent with the choice of the sign in front of the square root in (71), see Eq. (72). The point at which the equality holds in (74) correspond to the value of parameters for which the eigenvalue detaches from the lower edge of the support of the semicircle. For the values of the parameters q,ϵ and ϵ_0 that we are interested in, we find that the relevant solution, whenever it exists, equals to z_+. Given this solution, the isolated eigenvalue is obtained from:

$$\lambda_0(q,\epsilon,\epsilon_0)=z_+(q,\epsilon,\epsilon_0)-\sqrt{2}p\epsilon, \tag{75}$$

which is equivalent to Eq. (4) in the main text.

G. Additional results on the complexity

1. Quenched vs annealed average over the fixed minimum

The average over the disorder in Eq. (2) in the main text is conditioned to $\boldsymbol{\sigma}^0$, meaning:

$$\langle\cdot\rangle_0\equiv\left\langle\cdot\;\middle|\;\begin{Bmatrix}\mathbf{g}[\boldsymbol{\sigma}^0]=0,\\ h[\boldsymbol{\sigma}^0]=\sqrt{2N}\epsilon_0\end{Bmatrix}\right\rangle. \tag{76}$$

The resulting complexity does not depend explicitly on $\boldsymbol{\sigma}^0$, and can therefore be trivially averaged with respect to the flat measure over stationary points of a given energy ϵ_0, meaning that

$$\Sigma(\epsilon,q|\epsilon_0)=\lim_{N\to\infty}\frac{1}{N}\frac{1}{\mathcal{N}(\epsilon_0)}\int\mathcal{D}\boldsymbol{\sigma}^0\langle\log\mathcal{N}_{\boldsymbol{\sigma}^0}(\epsilon,q|\epsilon_0)\rangle_0, \tag{77}$$

where $\mathcal{D}\boldsymbol{\sigma}^0$ is the flat measure over stationary points with the right energy density,

$$\mathcal{D}\boldsymbol{\sigma}^0=d\boldsymbol{\sigma}^0\delta(h[\boldsymbol{\sigma}^0]-\sqrt{2N}\epsilon_0)\;\delta(\mathbf{g}[\boldsymbol{\sigma}^0]), \tag{78}$$

and $\mathcal{N}(\epsilon_0)$ is their total number. This corresponds to performing an *annealed* average over the stationary point $\boldsymbol{\sigma}^0$, which is in fact equal to the *quenched* average:

$$\lim_{N\to\infty}\frac{1}{N}\left\langle\frac{1}{\mathcal{N}(\epsilon_0)}\int\mathcal{D}\boldsymbol{\sigma}^0\log\mathcal{N}_{\boldsymbol{\sigma}^0}(\epsilon,q|\epsilon_0)\right\rangle, \tag{79}$$

where the average over the stationary points $\boldsymbol{\sigma}^0$ is performed *prior* to the disorder average. The reason for the equivalence is that stationary points of the unconstrained p-spin landscape are typically orthogonal to each others, and thus uncorrelated (this indeed also implies that the quenched and annealed complexity of the unconstrained p-spin coincide [4–6]). Indeed, the disorder average (79) can be computed as:

$$\begin{aligned}&\lim_{m\to0}\left\langle[\mathcal{N}(\epsilon_0)]^{m-1}\int\mathcal{D}\boldsymbol{\sigma}^0\log\mathcal{N}_{\boldsymbol{\sigma}^0}(\epsilon,q|\epsilon_0)\right\rangle=\\&\lim_{m\to0}\left\langle\int\mathcal{D}\boldsymbol{\sigma}^0\prod_{k=1}^{m-1}\mathcal{D}\boldsymbol{\tau}^k\log\mathcal{N}_{\boldsymbol{\sigma}^0}(\epsilon,q|\epsilon_0)\right\rangle,\end{aligned} \tag{80}$$

and similarly to (4) this equals to

$$\begin{aligned}\lim_{m\to0}\int d\boldsymbol{\sigma}^0\prod_{k=1}^{m-1}d\boldsymbol{\tau}^k\mathcal{E}_{\boldsymbol{\sigma}^0,\vec{\boldsymbol{\tau}}}(\epsilon_0)\,p_{\boldsymbol{\sigma}^0,\vec{\boldsymbol{\tau}}}(\mathbf{0},\epsilon_0)\cdot\\\cdot\left\langle\log\mathcal{N}_{\boldsymbol{\sigma}^0}(\epsilon,q|\epsilon_0)\right|\left\{\begin{smallmatrix}\mathbf{g}[\boldsymbol{\sigma}^0]=0,\mathbf{g}[\boldsymbol{\tau}^k]=0\\ h[\boldsymbol{\sigma}^0]=h[\boldsymbol{\tau}^k]=\sqrt{2N}\epsilon_0\end{smallmatrix}\right\}\Big\rangle,\end{aligned} \tag{81}$$

where $p_{\boldsymbol{\sigma}^0,\vec{\boldsymbol{\tau}}}(\mathbf{0},\epsilon_0)$ is now the joint distribution of gradients and energy densities of $\boldsymbol{\sigma}^0$ and of the $\boldsymbol{\tau}^k$, and $\mathcal{E}_{\boldsymbol{\sigma}^0,\vec{\boldsymbol{\tau}}}(\epsilon_0)$ the expectation value of the corresponding determinant. This expression can be parametrized in terms of the overlaps $z_{kl}=\boldsymbol{\tau}^k\cdot\boldsymbol{\tau}^l$, $z_{k0}=\boldsymbol{\tau}^k\cdot\boldsymbol{\sigma}^0$ and $z_{ka}=\boldsymbol{\tau}^k\cdot\boldsymbol{\sigma}^a$, in addition to the previously introduced overlaps q_{ab} and q, which is fixed. Performing first the saddle point over the overlaps between the $\boldsymbol{\tau}^k$ and $\boldsymbol{\sigma}^0$ one finds $z_{kl}=0=z_{k0}$ and $z_{ka}=0$; it follows that the disorder average in (81) becomes independent of the $\boldsymbol{\tau}^k$ and reduces to (76), and thus (81) reduces to (77).

2. The complexity at $q_M(\epsilon_0)$

The complexity is positive (implying that exponentially-many stationary points are present) only for $q_{\rm m}(\epsilon_0)\leq q\leq q_{\rm M}(\epsilon_0)$, where $q_{\rm m}(\epsilon_0)<0$. For each q in this range, the stationary points with energy below the threshold are distributed over an extensive spectrum of ϵ, which at $q_{\rm m}$ and $q_{\rm M}$ collapses to a single point. The lower boundary $q_{\rm m}(\epsilon_0)$ varies very little with ϵ_0, and $\bar{\epsilon}_{x=0}(q_{\rm m}|\epsilon_0)=\epsilon_{\rm th}$. At $q_{\rm M}(\epsilon_0)$, instead, two different situations are possible: for the smaller values of ϵ_0, the complexities $\Sigma(\epsilon,q|\epsilon_0)$ are increasing in the interval $\bar{\epsilon}_{x=0}(q|\epsilon_0)\leq\epsilon\leq\epsilon_{\rm th}$ for any q, and $q_{\rm M}(\epsilon_0)$ is the point at which $\bar{\epsilon}_{x=0}(q_{\rm M}|\epsilon_0)=\epsilon_{\rm th}$; for values of ϵ_0 very close to the threshold, instead, the complexity is no longer monotonic but has a local maximum within the interval, and $q_{\rm M}(\epsilon_0)$ is the latitude at which

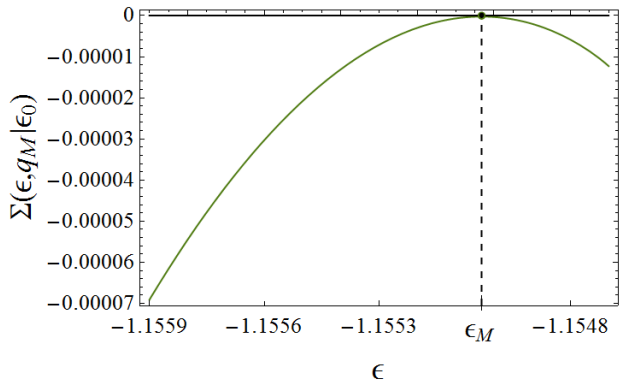


FIG. 2. Complexity for $\epsilon_0 = -1.158$ and $q = q_M(\epsilon_0) = 0.83683$. The local maximum is at $\epsilon_M(\epsilon_0) = -1.1550 < \epsilon_{\rm th}(\epsilon_0)$, where $\Sigma = 0$.

the local maximum touches zero, see Fig. 2. This implies that Σ is negative everywhere (included above the threshold) except at one precise value of energy density, $\epsilon_{\rm M}(\epsilon_0) \leq \epsilon_{\rm th}$, where $\Sigma = 0$. Note however that even for these ϵ_0 one can find barriers at energies up to $\epsilon_{\rm th}$, by focusing on small enough overlaps.

3. *The vanishing of the isolated eigenvalue.*

Fig. 1 in the main text shows that minima appear first at the overlap $q^*(\epsilon_0)$, and in a small interval of $q \lesssim q^*(\epsilon_0)$ they coexist with saddles: the higher-energy points are saddles and the lower energy ones are minima, separated by a family of *marginal* saddles (with one single zero mode), having finite complexity. At smaller values of q, all stationary points are minima. We now argue that, for any fixed ϵ_0 (we henceforth drop the dependence on ϵ_0), the iso-complexity curves $\bar\epsilon_x(q)$ satisfying

$$\Sigma(\bar\epsilon_x, q) \equiv x, \tag{82}$$

for those values of x for which they are non-monotonic below the threshold energy, have a local minimum at a point (say $q = q_x$) which is also the point at which the isolated eigenvalue vanishes,

$$\lambda_0(q_x, \bar\epsilon_x(q_x)) = 0. \tag{83}$$

Indeed, at $q = q_x$ and $\epsilon = \bar\epsilon_x(q_x)$ it holds simultaneously:

$$\begin{cases} \frac{\partial}{\partial q}\Sigma(\epsilon; q) & = 0 \\ \Sigma(\epsilon, q) & = x, \end{cases} \tag{84}$$

where the first equation follows from $d\Sigma(\bar\epsilon_x(q), q)/dq = 0$, using that $d\bar\epsilon_x(q)/dq = 0$. On the other hand, the isolated eigenvalue λ_0 vanishes whenever:

$$\mu(q, \epsilon) + \Delta^2 G_\sigma(\lambda_0) - \sqrt{2}p\epsilon = 0, \tag{85}$$

where we used that $\lambda_0 - \mu - \Delta^2 G_\sigma(\lambda_0) = 0$. Eq. (85) and the first of the Eqs. (84) are both second order equations for ϵ at fixed q, and substituting the explicit expressions for the constants it can be shown that they are proportional to each others, and thus admit identical solutions. This fixes two curves $\bar\epsilon_\pm(q)$, one of which can be selected by imposing the consistency with the sign in front of the square root of the resolvent. Imposing the condition $\Sigma(\bar\epsilon_\pm(q), q) = x$, one selects the point q_x.

4. *The energy barriers*

Fig. 3 shows the ϵ_0-dependence of the optimal barrier $\Delta\epsilon^*(\epsilon_0) = \epsilon^*(\epsilon_0) - \epsilon_0$ corresponding to the lowest-energy transition state, as well as of the closest barrier $\Delta\epsilon_M(\epsilon_0) = \epsilon_M(\epsilon_0) - \epsilon_0$ corresponding to the closest transition states at overlap q_M with the minimum. The barriers decrease with ϵ_0 and vanish at $\epsilon_0 = \epsilon_{\rm th}$. The energy difference $\Delta\epsilon^*(\epsilon_0)$ has an apparent linear behaviour with a slope larger than -1, similarly to what is found numerically in [7] (the slope -1 would correspond to the transition states having a constant energy independently of ϵ_0, as assumed for example in trap-models). The apparent linear behaviour is however just due to the proximity of $\epsilon^*(\epsilon_0)$ and $\epsilon_{\rm th}$; indeed, the plot of $\epsilon^*(\epsilon_0)$ in Fig. 3 in the main text clearly shows a more complicated, non-linear dependence of $\epsilon^*(\epsilon_0)$ on ϵ_0, that gives rise to small deviations from the linear behavior of the energy barriers. The closest barriers $\Delta\epsilon_M(\epsilon_0)$ have instead a strictly linear behavior for the smaller values of ϵ_0 (whenever $\epsilon_M = \epsilon_{\rm th}$).

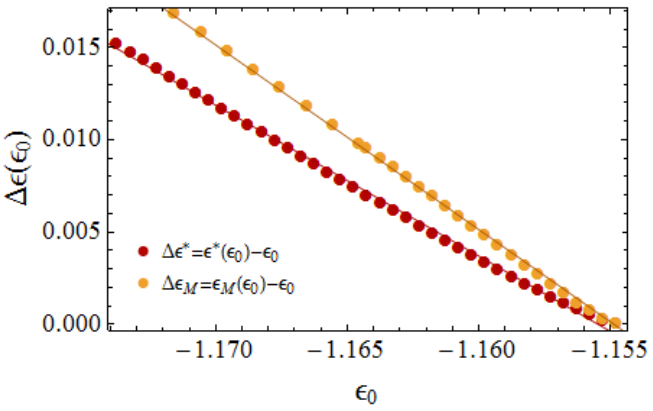
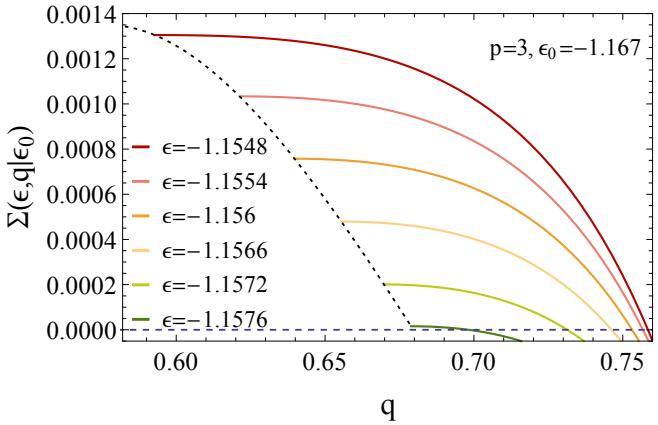


FIG. 3. Optimal barrier $\Delta\epsilon^*(\epsilon_0) = \epsilon^*(\epsilon_0) - \epsilon_0$ (red) and closest barrier $\Delta\epsilon_M(\epsilon_0) = \epsilon_M(\epsilon_0) - \epsilon_0$ (orange). The continuous lines are linear fits, with slope $-0.82 > -1$ and -1, respectively.

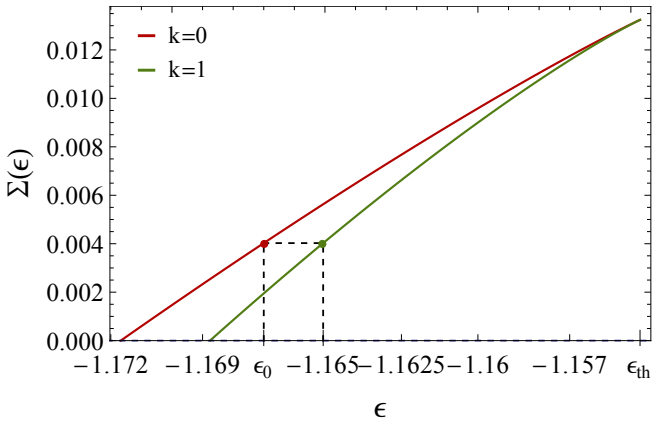
5. *Comparison with the unconstrained complexity of index-1 saddles*

Fig. 4 shows a comparison between the complexity of the saddles at fixed overlap with $\boldsymbol{\sigma}^0$ and the total complexity of the minima and order-1 saddles of the p-spin

landscape. It shows that the saddles found with our calculation are not the index-1 saddles having the same complexity as the family of minima to which $\boldsymbol{\sigma}^0$ belongs (*i.e.*, the minima with energy ϵ_0). Rather, the saddles found in the vicinity of $\boldsymbol{\sigma}^0$ have higher energy. We remind that these are the properties of the *typical* stationary points found at fixed q, *i.e.*, of the most numerous ones, counted by $\Sigma(\epsilon, q|\epsilon_0)$. Rarer points with different stability properties should be present at the same latitudes: to determine their complexity, however, one has to perform large deviation calculations by conditioning explicitly on their index. We leave this computation for future work.



(a)



(b)

FIG. 4. (a) Complexity of the saddles at overlap q from $\boldsymbol{\sigma}^0$. The curves reach their maximum at the overlaps corresponding to the marginal saddles with a single zero mode (they continue at smaller q with a decreasing branch, not plotted, corresponding to minima). The black dotted curve is the complexity of marginal saddles. (b) Unconstrained complexity of minima ($k = 0$) and index-1 saddles ($k = 1$). The red point identifies $\epsilon_0 = -1.167$, the green point the energy of the index-1 saddles equally numerous with respect to the minima at energy $\epsilon_0 = -1.167$.

H. Zero-temperature Franz-Parisi potential

In this last section, we report the saddle point equations obtained when computing the free energy of a system constrained to be at fixed overlap q from a given minimum of energy density $\epsilon_0 \geq \epsilon_{\rm gs}$. The computation is performed at zero temperature ($\beta' \to \infty$). The free energy is obtained as

$$F(q|\epsilon_0) = -\lim_{\beta'\to\infty}\lim_{n\to 0}\frac{1}{\beta'}\frac{S(q|\epsilon_0)}{n} \tag{86}$$

where $S(q|\epsilon_0)$ is such that:

$$\exp\left[NS(q|\epsilon_0)+o(N)\right] = \int d\mathbf{Q}\exp\left[NS(\mathbf{Q})+o(N)\right] = \left\langle \int d\mathbf{s}\exp\left(-\beta'\sum_{\alpha=1}^{n}\sum_{i_1<\cdots<i_p} J_{i_1\cdots i_p}s^\alpha_{i_1}\cdots s^\alpha_{i_p}\right)\right\rangle_0 . \tag{87}$$

The integral on the RHS is over configurations $\mathbf{s}$ on an hypersphere of radius $\sqrt{N}$ constrained to be at overlap q from a given minimum at energy density ϵ_0; the average is both on the random couplings, and on minima at energy ϵ_0. In order to average over minima we introduce additional m replicas and follow [8]: we select a given energy level by sampling from a Boltzmann measure at inverse temperature β, biased by the number of replicas m. The value of m is not optimized over as it should be done to obtain the equilibrium energy, but it is chosen in such a way to select the energy density ϵ_0 at will. In particular, to select minima we work in the limit $\beta \to \infty$ and $m \to 0$ such that βm is finite and chosen appropriately to select minima of energy ϵ_0. The resulting integral on the LHS of (87) is over $(n+m)$ replicas and therefore over $(n+m)\times(n+m)$ overlap matrices $\mathbf{Q}$, which parametrize the action as:

$$S(\mathbf{Q}) = \frac{\beta^2}{4}\sum_{a,b=1}^{m}Q^p_{a,b} + \frac{\beta'^2}{4}\sum_{\alpha,\beta=1}^{n}Q^p_{\alpha,\beta} + \frac{\beta\beta'}{2}\sum_{a,\alpha}Q^p_{a,\alpha} + \frac{1}{2}\log\det(\mathbf{Q}).$$

The overlap matrix $\mathbf{Q}$ has the following generic structure: it is replica-symmetric in the first $m\times m$ diagonal block, corresponding to the minimum of energy ϵ_0, and has entries equal to q_0 everywhere except from the diagonal elements, that are set to 1; it is either replica symmetric (or 1-step replica symmetry broken) in the second $n\times n$ diagonal block, which describes the system at fixed overlap q with the minimum, with 1 on the diagonal and q' anywhere else (or q'_1 in the $\mu\times\mu$ diagonal sub-blocks and q'_0 anywhere else); finally, the elements belonging to the $n\times m$ off-diagonal rectangles are all set equal to q. Below, we determine the saddle point equations and compute the corresponding action in the two cases.

1. The replica-symmetric case

In the RS case, the action reads as follows:

$$S_{RS} = \frac{1}{4}[\beta^2(m + m(m-1)q_0^p) + \beta'^2(n + n(n-1)q'^p) + \\ + 2mn\beta\beta' q^p] + \\ + \frac{1}{2}\{(m-1)\log(1-q_0) + (n-1)\log(1-q') + \\ + \log[(1 + (m-1)q_0)(1 + (n-1)q') - mnq^2]\}. \tag{88}$$

The saddle point equation for q_0 gives

$$\frac{\beta m}{2} p q_0^{p-1} = \frac{1}{\beta(1-q_0)} - \frac{\beta'(1-q'+nq')}{D} \tag{89}$$

with

$$D = \beta'(1-q'+nq')\beta(1-q_0+mq_0) - m\beta\beta' nq^2, \tag{90}$$

which for $n \to 0$ becomes:

$$\beta(1-q_0)\beta(1-q_0+mq_0) = \frac{2}{p}. \tag{91}$$

In the $\beta \to \infty$ limit, the product $\beta(1-q_0)$ remains finite and equals:

$$\beta(1-q_0) = \frac{1}{2}\left(-\beta m + \sqrt{(\beta m)^2 + \frac{8}{p}}\right), \tag{92}$$

and the energy ϵ_0 of the corresponding minimum can be written as:

$$\epsilon_0 = -\frac{1}{2}\left((p-1)\beta(1-q_0) + \frac{2}{p\beta(1-q_0)}\right). \tag{93}$$

These two equations fix βm as a function of the chosen ϵ_0.
The saddle point on q' gives instead:

$$\frac{\beta' n}{2} p q'^{p-1} = \frac{1}{\beta'(1-q')} - \frac{\beta(1-q_0+mq_0)}{D} \tag{94}$$

which in the $n \to 0$ limit reduces to

$$\frac{\beta'^2}{2} p q'^{p-1} = \frac{q'}{(1-q')^2}\left(1 - \frac{q^2 m}{q'(1-q_0+mq_0)}\right). \tag{95}$$

In the limit $\beta' \to \infty$, we have $q' \to 1$ with $\beta'(1-q')$ finite, fixed by the equation:

$$\beta'^2(1-q')^2 = \frac{2}{p}\left(1 - \frac{q^2\beta m}{\beta(1-q_0+mq_0)}\right). \tag{96}$$

This solution is stable for high values of the overlap q and until

$$\beta'^2(1-q')^2 = \frac{2}{p(p-1)}, \tag{97}$$

where the replicon eigenvalue of the hessian corresponding to this solution vanishes [4].

2. The 1-step replica symmetry broken case

In this case the **Q** matrix has a 1RSB structure in the $n \times n$ block, with q_1' on the diagonal $\mu \times \mu$ blocks and q_0' in the rest of the matrix, except from the diagonal which is equal to 1. The action reads:

$$S_{1RSB} = \\ \frac{1}{4}[\beta^2(m + m(m-1)q_0^p) + \\ + \beta'^2(n + n(\mu-1)q_1'^p + n(n-\mu)q_0'^p) + 2mn\beta\beta' q^p] + \\ + \frac{1}{2}\left[(m-1)\log(1-q_0) + n\left(1-\frac{1}{\mu}\right)\log(1-q_1') + \right. \\ + \left(\frac{n}{\mu} - 1\right)\log(1 - q_1' + \mu(q_1' - q_0')) + \\ \left. + \log[(1-q_0+mq_0)(1-q_1'+\mu(q_1'-q_0')+nq_0') - mnq^2]\right]. \tag{98}$$

For $q = 0$, this action reduces to the sum of one RS and one 1RSB actions with, respectively, inverse temperature β and m replicas, and inverse temperature β' and a 1RSB structure with parameters n and μ: $S_{1RSB}(q=0) = S_{RS}(\beta, m) + S_{1RSB}(\beta', n, \mu)$. For arbitrary q, the saddle point on q_0 in the $n \to 0$ limit does not change with respect to the previous case, Eq. (91). The saddle point equation for q_0' and a combination of it with the equation for q_1' give:

$$\frac{p}{2} q_0'^{p-1} = \frac{1}{\beta'^2[1-q_1'+\mu(q_1'-q_0')]^2}\left(q_0' - \frac{m}{1-q_0+mq_0}q^2\right) \tag{99}$$

and

$$\frac{p}{2}(q_1'^{p-1} - q_0'^{p-1}) = \frac{q_1' - q_0'}{\beta'^2[1-q_1'+\mu(q_1'-q_0')](1-q_1')}. \tag{100}$$

Finally, the saddle point on μ gives

$$0 = \frac{q_1'^p - q_0'^p}{2} + \frac{1}{\beta'^2\mu^2}\log\left(\frac{1-q_1'}{1-q_1'+\mu(q_1'-q_0')}\right) + \\ + \frac{q_1' - q_0'}{\beta'[1-q_1'+\mu(q_1'-q_0')]} \times \\ \times \left[\frac{1}{\mu\beta'} - \frac{1}{\beta'[1-q_1'+\mu(q_1'-q_0')]}\left(q_0' - \frac{mq^2}{1-q_0+mq_0}\right)\right]. \tag{101}$$

For βm and q fixed, in the limit $\beta' \to \infty$ these equations have non-trivial solutions for $\beta'\mu$ finite (hence $\mu \to 0$), $\beta'(1-q_1')$ finite (hence $q_1' \to 1$), and $q_1' - q_0'$ finite. For $q = 0$, the usual 1RSB saddle point equations of the p-spin are recovered.
In both the RS and 1RSB case (Eq. (88) and (98) respectively), for any fixed βm the zero temperature FP potential, *i.e.* the minimal energy at fixed overlap from a typical minimum at energy density ϵ_0, is obtained by taking the derivative with respect to n of the RS and

1RSB action as follows

$$\epsilon_{\rm FP}(q|\epsilon_0) = -\lim_{\beta'\to\infty}\lim_{n\to 0}\frac{1}{\beta'}\frac{\partial S(q|\epsilon_0)}{\partial n}, \tag{102}$$

evaluating it in the $\beta\to\infty, m\to 0$ limit with βm fixed. Using that $q_1', q'\to 1$, we find in the RS case:

$$\begin{aligned}\epsilon_{\rm FP}^{(RS)} =& -\frac{1}{2}\left(\beta m q^p + \frac{p}{2}\beta'(1-q')\right)\\ &-\frac{1}{2}\frac{\beta(1-q_0)+\beta m(1-q^2)}{\beta'(1-q')(\beta(1-q_0)+\beta m)},\end{aligned} \tag{103}$$

while in the 1RSB case we have:

$$\begin{aligned}\epsilon_{\rm FP}^{(1RSB)} =& -\frac{1}{2}\left[\beta m q^p + \frac{1}{2}\left(p\beta'(1-q_1') + \beta'\mu(1-q_0'^p)\right)\right]\\ &-\frac{1}{2}\frac{q_0'\left(\beta(1-q_0)+\beta m\right)-\beta m q^2}{\left(\beta'(1-q_1')+\beta'\mu(1-q_0')\right)\left(\beta(1-q_0)+\beta m\right)}\\ &-\frac{1}{2\beta'\mu}\log\left(\frac{\beta'(1-q_1')+\beta'\mu(1-q_0')}{\beta'(1-q_1')}\right),\end{aligned} \tag{104}$$

which reduces to the RS case when $q_0' = q_1' \to 1$. The expression (103) holds at high-enough q, until the condition (97) is met, while (104) holds at the smaller values of q. Substituting in these expression the solutions of the corresponding saddle point equations above, we obtain the result shown in Fig.4 of the main text. We remark that the FP has always a global minimum at $q = 0$, which corresponds to the second configuration being at equilibrium at zero-temperature, independently from the fixed minimum of energy ϵ_0; indeed, the corresponding energy (104) equals to the ground-state energy of the unperturbed p-spin. The local minimum of the zero-temperature FP potential is always attained at $q = 1$: this corresponds to the second configuration being inside the state identified by the minimum of energy ϵ_0, which at zero-temperature reduces to a single configuration (hence, the corresponding overlap equals to one). Finally, at the local maximum, where the potential coincides with $\bar{\epsilon}_{x=0}$, it holds $q_0' = q^2$, consistently with the saddle point solution for the overlap q_1 between the replicas in the Kac-Rice calculation (at that value of q, one finds indeed $q_0' = q_1$).

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