Supplemental Materials: Remanent magnetization: signature of Many-Body Localization in quantum antiferromagnets

DERIVATION OF EQ. (4)

The operator in Eq. (3) in the main text is rewritten as

$$\sum_{\alpha} P_\alpha \sigma_j^z P_\alpha = \sum_{i_1=\pm1}\sum_{i_2=\pm1}\cdots\sum_{i_L=\pm1}\prod_{k=1}^{L} P(i_k)\,\sigma_j^z\prod_{k=1}^{L} P(i_k), \tag{1}$$

where we introduced the projectors:

$$P(i_k) \equiv \frac{1+i_k I_k}{2}. \tag{2}$$

To derive Eq. (4) in the main text, we make use of the following operator identities:

$$AB = BA + [A,B]\,, \qquad \left[A, \prod_{k=1}^{L} B_k\right] = \sum_{k_1=1}^{L}\left(\prod_{k=1}^{k_1-1} B_k\right)[A, B_{k_1}]\left(\prod_{k=k_1+1}^{L} B_k\right). \tag{3}$$

For

$$A^{(1)} = \sigma_j^z, \quad B^{(1)} = \prod_{k=1}^{L} B_k = \prod_{k=1}^{L} P(i_k), \tag{4}$$

the above identities imply

$$\prod_{k=1}^{L} P(i_k)\,\sigma_j^z\prod_{k=1}^{L} P(i_k) = \prod_{k=1}^{L} P(i_k)\left(\sigma_j^z + \sum_{k_1=1}^{L}\left[\sigma_j^z, \frac{i_{k_1}I_{k_1}}{2}\right]\prod_{k=k_1+1}^{L} P(i_k)\right). \tag{5}$$

Applying (3) once more with

$$A^{(2)} = \left[\sigma_j^z, \frac{i_{k_1}I_{k_1}}{2}\right], \quad B^{(2)} = \prod_{k=k_1+1}^{L} P(i_k) \tag{6}$$

yields

$$\prod_{k=1}^{L} P(i_k)\,\left[\sigma_j^z, \frac{i_{k_1}I_{k_1}}{2}\right]\prod_{k=k_1+1}^{L} P(i_k) = \prod_{k=1}^{L} P(i_k)\left(\left[\sigma_j^z, \frac{i_{k_1}I_{k_1}}{2}\right] + \sum_{k_2=k_1+1}^{L}\left[\left[\sigma_j^z, \frac{i_{k_1}I_{k_1}}{2}\right], \frac{i_{k_2}I_{k_2}}{2}\right]\prod_{k=k_2+1}^{L} P(i_k)\right). \tag{7}$$

Further iteration with

$$A^{(n)} = \left[\left[\left[\sigma_j^z, \frac{i_{k_1}I_{k_1}}{2}\right], \cdots\right], \frac{i_{k_{n-1}}I_{k_{n-1}}}{2}\right], \quad B^{(n)} = \prod_{k=k_{n-1}+1}^{L} P(i_k) \tag{8}$$

finally leads to

$$\prod_{k=1}^{L} P(i_k)\,\sigma_j^z\prod_{k=1}^{L} P(i_k) = \prod_{k=1}^{L} P(i_k)\left(\sigma_j^z + \sum_{N=1}^{L}\sum_{k_N>\cdots>k_1}\left[\left[\left[\sigma_j^z, \frac{i_{k_1}I_{k_1}}{2}\right], \cdots\right], \frac{i_{k_N}I_{k_N}}{2}\right]\right). \tag{9}$$

The identity (4) is established using that $i_k \in \{\pm1\}$ and that

$$\sum_{i_k=\pm1} P(i_k) = \mathbb{1}. \tag{10}$$

EXPLICIT EXPRESSION FOR CONSERVED QUANTITIES

As derived in [27], the formal expression for the terms in the perturbative expansion in Eq. (6) in the main text reads:

$$\delta I_k^{(n)} = i \lim_{\eta\to 0} \int_0^\infty d\tau e^{-\eta\tau} e^{iH_0\tau} \left[H_1, \delta I_k^{(n-1)}\right] e^{-iH_0\tau} + \Delta I_k^{(n)}, \tag{11}$$

where in this case

$$\begin{aligned} H_0 &= \sum_i \left(h_i \sigma_i^z - J_z \sigma_i^z \sigma_{i+1}^z\right), \\ H_1 &= -\sum_i J_x \sigma_i^x \sigma_{i+1}^x. \end{aligned} \tag{12}$$

The operator $\Delta I_k^{(n)}$ in (11) is a suitable polynomial in the σ_i^z, which ensures that $I_k^2 = \mathbb{1}$ is satisfied at the given order in H_1: neglecting the $\Delta I_k^{(n)}$ at any order leads to a modified operator that is still conserved, although it does not have binary spectrum ± 1. At first order $n = 1$, one finds that $\Delta I_k^{(1)} = 0$, while $\delta I_k^{(1)}$ is given in Eq. (7) in the main text.

We now discuss how the perturbative expansion needs to be modified in presence of resonances, in order to obtain the operators $\tilde{I}_k, \tilde{I}_{k+1}$ in the main text. Let $k, k+1$ be the sites giving rise to a first order resonance, i.e., to a small denominator for a particular choice of τ, ρ in Eq. (9) in the main text. We aim at finding the set of spin operators that is conserved by the reduced Hamiltonian

$$H^{(k)} = H_0 - J_x \sigma_k^x \sigma_{k+1}^x = H_0 - J_x \left(\tilde{O}_+^{(k)} + \tilde{\Delta}_+^{(k)}\right) \equiv H_0 + H_1^{(k)}, \tag{13}$$

where we introduced $\tilde{O}_\pm^{(k)} = \sigma_k^+ \sigma_{k+1}^- \pm \sigma_k^- \sigma_{k+1}^+$, and $\tilde{\Delta}_\pm^{(k)} = \sigma_k^+ \sigma_{k+1}^+ \pm \sigma_k^- \sigma_{k+1}^-$. The first-order term in the perturbative expansion,

$$\hat{I}_k = \sigma_k^z + \delta I_k^{(1)} = \sigma_k^z + \sum_{\rho,\tau \pm 1} \left(A_{\rho\tau}^{(k)} O_{\rho\tau}^{(k)} + B_{\rho\tau}^{(k)} \Delta_{\rho\tau}^{(k)}\right), \tag{14}$$

is exactly conserved by (13): This can be deduced from (11) setting $H_1 \to H_1^{(k)}$ and $\Delta I_k^{(n)} = 0 \;\; \forall n$, noticing that $\left[H_1^{(k)}, \delta I_k^{(1)}\right] = 0$ and thus that the perturbative expansion terminates at the first order. However, the operator does not square to the identity: To impose the binarity of the spectrum, it is necessary to introduce an operator $\Delta I_k^{(2)}$ canceling the terms $\hat{I}_k^2 - \mathbb{1}$ which are of second order in J_x. The resulting operator $\hat{I}_k + \Delta I_k^{(2)}$ will no longer commute with $H_1^{(k)}$, giving rise through (11) to a full perturbative series that needs to be resummed to get the $\tilde{I}_k, \tilde{I}_{k+1}$.

In the case of the Hamiltonian (13), it is possible to circumvent the resummation by directly guessing the form of the operators $\tilde{I}_k, \tilde{I}_{k+1}$. Indeed, the terms in $\hat{I}_k^2 - \mathbb{1}$ are proportional to:

$$\begin{aligned} \left(\sigma_k^+ \sigma_{k+1}^+ + h.c.\right)^2 &= \frac{1 + \sigma_k^z \sigma_{k+1}^z}{2} \equiv \tilde{J}_+^{(k)}, \\ \left(\sigma_k^+ \sigma_{k+1}^- + h.c.\right)^2 &\equiv \frac{1 - \sigma_k^z \sigma_{k+1}^z}{2} = \tilde{K}_+^{(k)}, \end{aligned} \tag{15}$$

where we defined

$$\tilde{J}_\pm^{(k)} = P_{1,1}^{(k)} \pm P_{-1,-1}^{(k)}, \quad \tilde{K}_+^{(k)} = P_{1,-1}^{(k)} \pm P_{-1,1}^{(k)}, \tag{16}$$

and

$$P_{\rho,\tau}^{(k)} = \frac{1 + \rho\,\sigma_k^z}{2} \, \frac{1 + \tau\,\sigma_{k+1}^z}{2}. \tag{17}$$

Exactly the same operators as in (15) are generated by the following anticommutators:

$$\begin{aligned} \frac{1}{2}\left\{\sigma_k^z, \tilde{K}_-^{(k)}\right\} &= \frac{1}{2}\left\{\sigma_k^z, \frac{\sigma_k^z - \sigma_{k+1}^z}{2}\right\} = P_{1,-1}^{(k)} + P_{-1,1}^{(k)}, \\ \frac{1}{2}\left\{\sigma_k^z, \tilde{J}_-^{(k)}\right\} &= \frac{1}{2}\left\{\sigma_k^z, \frac{\sigma_k^z + \sigma_{k+1}^z}{2}\right\} = P_{1,1}^{(k)} + P_{-1,-1}^{(k)}. \end{aligned} \tag{18}$$

This suggests to add to the $\hat{I}_k$ some operators proportional to $\tilde{K}_-^{(k)}$ and $\tilde{J}_-^{(k)}$: since the latter anticommute with $\tilde{O}_+^{(k)}$ and $\tilde{\Delta}_+^{(k)}$, no additional operator is produced when squaring the sum of $\hat{I}_k$ with the newly added operators; similarly, no additional operator appears when imposing conservation, given that the commutators $\left[\tilde{O}_+^{(k)}, \tilde{K}_-^{(k)}\right]$ and $\left[\tilde{\Delta}_+^{(k)}, \tilde{J}_-^{(k)}\right]$ are proportional to the commutators $\left[\sigma_k^z, \tilde{O}_+^{(k)}\right]$ and $\left[\sigma_k^z, \tilde{\Delta}_+^{(k)}\right]$, while all other commutators are zero.

Based on these observations, we introduce:

$$\tilde{I}_k = \sigma_k^z + \sum_{\rho\tau=\pm1} \left(\tilde{A}_{\rho\tau}^{(k)} O_{\rho\tau}^{(k)} + C_{\rho\tau}^{(k)} K_{\rho\tau}^{(k)}\right) + \sum_{\rho\tau=\pm1} \left(\tilde{B}_{\rho\tau}^{(k)} \Delta_{\rho\tau}^{(k)} + D_{\rho\tau}^{(k)} J_{\rho\tau}^{(k)}\right), \tag{19}$$

with

$$\begin{aligned} K_{\rho\tau}^{(k)} &= \frac{1+\rho\,\sigma_{k-1}^z}{2}\left[P_{1,-1}^{(k)} - P_{-1,1}^{(k)}\right]\frac{1+\tau\,\sigma_{k+2}^z}{2}, \\ J_{\rho\tau}^{(k)} &= \frac{1+\rho\,\sigma_{k-1}^z}{2}\left[P_{1,1}^{(k)} - P_{-1,-1}^{(k)}\right]\frac{1+\tau\,\sigma_{k+2}^z}{2}, \end{aligned} \tag{20}$$

and with coefficients that are of infinite order in the resonant coupling J_x. Imposing $\left[\tilde{I}_k, H^{(k)}\right] = 0$ and collecting the coefficient in front of each operator we find:

$$\begin{aligned} \tilde{A}_{\rho\tau}^{(k)}\left(h_k - h_{k+1} + J_z(\tau-\rho)\right) + J_x\left(1 + C_{\rho\tau}^{(k)}\right) &= 0 \\ \tilde{B}_{\rho\tau}^{(k)}\left(h_k + h_{k+1} - J_z(\tau+\rho)\right) + J_x\left(1 + D_{\rho\tau}^{(k)}\right) &= 0, \end{aligned} \tag{21}$$

from which Eqs. (9) in the main text are recovered for $C_{\rho\tau}^{(k)} = 0 = D_{\rho\tau}^{(k)}$. Using (15) and (18) we obtain that $\tilde{I}_k^2 = \mathbb{1}$ is satisfied provided

$$\begin{aligned} \left(\tilde{A}_{\rho\tau}^{(k)}\right)^2 + \left(C_{\rho\tau}^{(k)}\right)^2 + 2C_{\rho\tau}^{(k)} &= 0, \\ \left(\tilde{B}_{\rho\tau}^{(k)}\right)^2 + \left(D_{\rho\tau}^{(k)}\right)^2 + 2D_{\rho\tau}^{(k)} &= 0, \end{aligned} \tag{22}$$

for each choice of $\tau, \rho = \pm1$. It can be checked that Eqs.(21), (22) admit the solutions:

$$\begin{aligned} \tilde{A}_{\rho\tau}^{(k)} &= -\frac{J_x}{\left(\left[h_k - h_{k+1} + J_z(\tau-\rho)\right]^2 + J_x^2\right)^{1/2}}, \\ C_{\rho\tau}^{(k)} &= -1 + \frac{h_k - h_{k+1} + J_z(\tau-\rho)}{\left(\left[h_k - h_{k+1} + J_z(\tau-\rho)\right]^2 + J_x^2\right)^{1/2}}, \\ \tilde{B}_{\rho\tau}^{(k)} &= -\frac{J_x}{\left(\left[h_k + h_{k+1} - J_z(\tau+\rho)\right]^2 + J_x^2\right)^{1/2}}, \\ D_{\rho\tau}^{(k)} &= -1 + \frac{h_k + h_{k+1} - J_z(\tau+\rho)}{\left(\left[h_k + h_{k+1} - J_z(\tau+\rho)\right]^2 + J_x^2\right)^{1/2}}. \end{aligned} \tag{23}$$

The coefficient $\tilde{B}_{\rho\tau}^{(k)}$ is the one exploited in the main text. Similar expressions are obtained for the operator $\tilde{I}_{k+1}$.

SCALING OF THE IMBALANCE FOR NON-INTERACTING FERMIONS

For large J/h- behavior of $\hat{\mathcal{I}}$ in the fermionic, single-particle case can be understood by setting

$$\phi_\alpha^2(k) = \frac{x_k^\alpha}{\xi} e^{-\frac{|k-r_\alpha|}{\xi}}, \tag{24}$$

where r_α denotes the localization center of the single particle wavefunction ϕ_α, ξ its localization length (we are neglecting its energy dependence), and the x_k^α are positive random variables of $O(1)$ that capture the fluctuations of the squared amplitudes under the exponentially decaying envelope. Partitioning the chain into segments of length $l = \lfloor \xi \rfloor$ and approximating the x_k^α as uncorrelated variables we obtain:

$$\hat{\mathcal{I}} \approx \frac{1}{L}\sum_{\alpha=1}^{L}\left(\sum_{R=1}^{L/l}(-1)^{R\,l}\,\frac{e^{-|R-R_\alpha|}}{\xi}\sum_{k=(R-1)l}^{R\,l}(-1)^k x_k^\alpha\right)^2 \approx \frac{1}{L}\sum_{\alpha=1}^{L}\left(\sum_{R=1}^{L/l}(-1)^{R\,l}\,\frac{e^{-|R-R_\alpha|}}{\sqrt{\xi}}\right)^2 \sim \frac{c}{\xi} \sim c\left(\frac{h}{J}\right)^2, \tag{25}$$

where R_α is the block containing the localization center r_α, and we have used that in the weak-disorder regime $\xi \sim (J/h)^2$ [55].

[27] V. Ros, M. Müller, and A. Scardicchio, Nucl. Phys. B **891**, 420 (2015).
[55] L. Molinari, J. Phys. A Math. Gen. **25**, 513 (1992).