Supplemental Material
Generalized Lotka-Volterra equations with random, non-reciprocal interactions: the typical number of equilibria

We report in the following the main steps to obtain the quantity $\bar{\mathcal{A}}(\mathbf{x};\phi)$ appearing in Eq. (9) in the main text. Moreover, we discuss additional results on the unbounded phase and on the vanishing of the total complexity, which are mentioned in the main text. For a more detailed exposition of the formalism underlying this calculation, we refer the reader to Ref. [1].

The Kac-Rice formula for the moments. The Kac-Rice formalism is a framework that allows one to characterize the number of solutions of dynamical equations containing randomness: in particular, given that the number of solutions is itself a random variable, the formalism gives a recipe to determine the moments of this random variable. For an introduction to the formalism and to its application to the high-dimensional setting, see [2, 3] and references therein. This formalism provides us with an expression for the moments of the number of equilibria at fixed diversity, denoted with $\mathcal{N}_S(\phi)$ in the main text. To compute the n-th moment of this random variable, we need to introduce n different configurations $\vec{N}^a$ of the ecosystem (with $a = 1, \cdots n$), which we refer to as *replicas*. Each $\vec{N}^a$ represents a realization of the ecosystem at fixed values of the rand interaction terms a_{ij}. We let $\mathbf{N} = (\vec{N}^1, \cdots, \vec{N}^n)$ denote the concatenation of configurations of all replicas. In each configuration $\vec{N}^a$, some species will be present ($N_i^a > 0$) while some others will be absent ($N_i^a = 0$). We let $I_a = I(\vec{N}^a)$ be the index set collecting the indices of the species that are present in the configuration $\vec{N}^a$. Since we are interested in counting the equilibria having fixed diversity ϕ, we enforce that $|I_a| = S\phi$ for all a. We introduce the vectors of growth rates or forces $\vec{F}^a = \vec{F}(\vec{N}^a)$ and $\mathbf{F}(\mathbf{N}) = (\vec{F}^1, \cdots, \vec{F}^n)$. Let $\mathbf{f}$ denote the value taken by this random vector, and $\mathcal{P}_{\mathbf{N}}(\mathbf{f})$ the joint distribution of the S-dimensional vectors $\vec{F}^a$ evaluated at $\vec{f}^a$,

$$\mathcal{P}^{(n)}_{\mathbf{N}}(\mathbf{f}) = \int \prod_{i,j=1}^{S} da_{ij} \mathbb{P}(\{a_{ij}\}_{ij})\, \delta\left(\mathbf{F}(\mathbf{N}) - \mathbf{f}\right). \tag{1}$$

We also introduce the following conditional expectation value:

$$\mathcal{D}^{(n)}_{\mathbf{N}}(\mathbf{f}) = \left\langle \left(\prod_{a=1}^{n} \left| \det \left(\frac{\delta F_i^a}{dN_j^a} \right)_{i,j\in I_a} \right| \right) \;\Big|\; \mathbf{F}(\mathbf{N}) = \mathbf{f} \right\rangle. \tag{2}$$

The latter is the expectation of the product of the absolute values of n determinants of the $S\phi \times S\phi$ matrices of derivatives of the components of $\mathbf{F}$, conditioned to $\mathbf{F}$ itself taking value $\mathbf{f}$. The Kac-Rice formula for the n-th moment of the number $\mathcal{N}_S(\phi)$ of uninvadable equilibria reads:

$$\langle \mathcal{N}^n(\phi) \rangle = \sum_{\substack{I_1 \\ |I_1|=S\phi}} \cdots \sum_{\substack{I_n \\ |I_n|=S\phi}} \prod_{a=1}^{n} \int d\vec{N}^a\, d\vec{f}^a \prod_{i\in I_a} \theta(N_i^a)\, \delta(f_i^a) \prod_{i\notin I_a} \delta(N_i^a)\theta(-f_i^a) \mathcal{D}^{(n)}_{\mathbf{N}}(\mathbf{f})\, \mathcal{P}^{(n)}_{\mathbf{N}}(\mathbf{f}). \tag{3}$$

We now briefly summarize how to determine the behaviour of the moments (3) for generic values of n to leading exponential order in S, and how to extract the quenched (and annealed) complexity from it.

The order parameters and the complexity. By performing the averages over the random interactions a_{ij}, one sees that the quantities $\mathcal{D}^{(n)}_{\mathbf{N}}(\mathbf{f})$ and $\mathcal{P}^{(n)}_{\mathbf{N}}(\mathbf{f})$ in (3) depend on the vectors $\vec{N}^a$ and $\vec{f}^a$ only through their scalar products. For $a, b = 1, \cdots, n$ we can therefore introduce a set of *order parameters* defined as follows:

$$Sq_{ab} = \vec{N}^a \cdot \vec{N}^b, \quad S\xi_{ab} = \vec{f}^a \cdot \vec{f}^b, \quad Sz_{ab} = \vec{N}^a \cdot \vec{f}^b, \quad Sm_a = \vec{N}^a \cdot \vec{1}, \quad Sp_a = \vec{f}^a \cdot \vec{1}, \tag{4}$$

where $\vec{1} = (1, \cdots, 1)^T$ is an S-dimensional vector with all entries equal to one. It follows that the integration over $\vec{N}^a, \vec{f}^a$ in (3) can be replaced by an integration over the order parameters, with the appropriate change of variables. The calculation proceeds in a few steps that we briefly summarize. First, the order parameters are introduced in (3) by means of the identities:

$$1 = \int dq_{ab}\, \delta\left(\frac{\vec{N}^a \cdot \vec{N}^b}{S} - q_{ab} \right) = S \int dq_{ab} \int \frac{d\hat{q}_{ab}}{2\pi} e^{i\hat{q}_{ab}\left(\vec{N}^a\cdot\vec{N}^b - Sq_{ab}\right)}, \tag{5}$$

where the auxiliary variables $\hat{q}_{ab}$ are *conjugate parameters* (and similarly for the other order parameters in (4)). Then, we make use of the assumption that the order parameters are symmetric with respect to permutations of the replicas, which implies that:

$$q_{ab} = \delta_{ab} q_1 + (1-\delta_{ab}) q_0, \quad \xi_{ab} = \delta_{ab}\xi_1 + (1-\delta_{ab})\xi_0, \quad z_{ab} = (1-\delta_{ab}) z, \quad m^a = m, \quad p^a = p, \tag{6}$$

and similarly for the conjugate ones. Let then $\mathbf{x} = (m, p, q_1, q_0, \xi_1, \xi_0)$ denote the collection of all of these order parameters, and $\hat{\mathbf{x}} = (\hat{m}, \hat{p}, \hat{q}_1, \hat{q}_0, \hat{\xi}_1, \hat{\xi}_0)$ the collection of the conjugate ones. Performing the integration over $\vec{N}^a, \vec{f}^a$ at fixed values of $\mathbf{x}, \hat{\mathbf{x}}$ and performing an expansion of the resulting expressions for large S, one then obtains the following integral representation for the moments:

$$\langle \mathcal{N}^n(\phi) \rangle = \int d\mathbf{x}\, i d\hat{\mathbf{x}}\, e^{S\,\mathcal{A}_n(\mathbf{x},\hat{\mathbf{x}},\phi) + o(S)}, \tag{7}$$

where the function $\mathcal{A}_n(\mathbf{x},\hat{\mathbf{x}},\phi)$ depends only on the order parameters and on the conjugate parameters, as well as on the number n of replicas. Given that S is large, the leading order contribution to the moments can be determined by means of a saddle point approximation, by evaluating $\mathcal{A}_n(\mathbf{x},\hat{\mathbf{x}},\phi)$ at the stationary point $\mathbf{x}^*, \hat{\mathbf{x}}^*$ which maximizes it. This can be done in principle for arbitrary values of n. We recall that the *annealed* complexity is obtained taking the logarithm of (7) with $n=1$, while the *quenched* complexity is obtained taking the limit $n \to 0$ according to Eq. (6). By choosing $n=1$, we obtain:

$$\mathcal{A}_1(\mathbf{x},\hat{\mathbf{x}},\phi) = \surd_1(\mathbf{x}) + \lceil(\phi) + \left(\hat{q}_1 q_1 + \hat{\xi}_1 \xi_1 + \hat{m} m + \hat{p} p + \hat{\phi}\phi\right) + \mathcal{J}_1(\hat{\mathbf{x}}), \tag{8}$$

with

$$\surd_1(\mathbf{x}) = -\frac{1}{2\sigma^2 q_1^2}\left[(\kappa-\mu m)^2\left(q_1 - \frac{\gamma m^2}{1+\gamma}\right) - 2(\kappa-\mu m) q_1\left(p + \frac{m}{1+\gamma}\right) + \xi_1 q_1\right] - \frac{1}{2}\log(2\pi\sigma^2 q_1) - \frac{1}{2\sigma^2(1+\gamma)}, \tag{9}$$

$$\mathcal{J}_1(\hat{\mathbf{x}}) = \log\left[\frac{1}{2}\sqrt{\frac{\pi}{\hat{\xi}_1}}\, e^{\frac{\hat{p}^2}{4\hat{\xi}_1}}\,\mathrm{Erfc}\left(-\frac{\hat{p}}{2\sqrt{\hat{\xi}_1}}\right) + \frac{e^{-\hat{\phi}}}{2}\sqrt{\frac{\pi}{\hat{q}_1}}\, e^{\frac{\hat{m}^2}{4\hat{q}_1}}\,\mathrm{Erfc}\left(\frac{\hat{m}}{2\sqrt{\hat{q}_1}}\right)\right], \tag{10}$$

and

$$\lceil(\phi) = \frac{\phi}{\pi}\int_{-1}^{1} dx \int_0^{\sqrt{1-x^2}} dy \log\left\{\left[\sigma\sqrt{\phi}(1+\gamma)x + 1\right]^2 + \sigma^2\phi(1-\gamma)^2 y^2\right\}. \tag{11}$$

This double integral can be evaluated explicitly, and one finds:

$$\lceil(\phi) = \begin{cases} \frac{1}{4\gamma\sigma^2}\left(1 - \sqrt{1-4\gamma\sigma^2\phi}\right) + \phi\log\left(1 + \sqrt{1-4\gamma\sigma^2\phi}\right) - \phi\left(\frac{1}{2} + \log 2\right) & \phi \le \phi_{\mathrm{May}} = \frac{1}{\sigma^2(1+\gamma)^2} \\ \frac{1}{2\sigma^2}\frac{1}{1+\gamma} - \frac{\phi}{2} + \frac{\phi}{2}\log(\sigma^2\phi) & \phi > \phi_{\mathrm{May}} = \frac{1}{\sigma^2(1+\gamma)^2}. \end{cases} \tag{12}$$

As expected, the functional (8) does not depend on q_0, ξ_0, z and on the associated conjugate parameters, that have a meaning only whenever more than one replica is present ($n > 1$). We consider now the case $n \to 0$, relevant to determine the quenched complexity. It can be shown that $\mathcal{A}_n(\mathbf{x},\hat{\mathbf{x}},\phi)$ admits the expansion:

$$\mathcal{A}_n(\mathbf{x},\hat{\mathbf{x}},\phi) = n\,\bar{\mathcal{A}}(\mathbf{x},\hat{\mathbf{x}},\phi) + o(n). \tag{13}$$

Explicitly, for general γ we find:

$$\bar{\mathcal{A}}(\mathbf{x},\hat{\mathbf{x}},\phi) = \bar{\surd}(\mathbf{x}) + \lceil(\phi) + \hat{q}_1 q_1 + \hat{\xi}_1\xi_1 + \hat{m} m + \hat{p} p + \hat{\phi}\phi - \frac{1}{2}\left(\hat{q}_0 q_0 + \hat{\xi}_0\xi_0\right) - \hat{z} z + \bar{\mathcal{J}}(\hat{\mathbf{x}}), \tag{14}$$

where $\lceil(\phi)$ is as above, while

$$\begin{aligned}\bar{\surd}(\mathbf{x}) = {} & \frac{(\kappa-\mu m)}{\sigma^2(1+\gamma)}\frac{m(q_1 - q_0 + z\gamma)}{(q_1-q_0)^2} + \frac{(\kappa-\mu m)}{\sigma^2}\frac{p}{(q_1-q_0)} - \frac{\gamma}{2\sigma^2(1+\gamma)}\frac{z^2(q_1+q_0)}{(q_1-q_0)^3} - \frac{\xi_1}{2\sigma^2(q_1-q_0)} \\ & - \frac{q_0(\xi_0-\xi_1)}{2\sigma^2(q_1-q_0)^2} - \frac{1}{2\sigma^2(1+\gamma)}\left[1 + \frac{2q_0 z}{(q_1-q_0)^2}\right] - \frac{1}{2\sigma^2}\frac{(\kappa-\mu m)^2}{q_1-q_0} - \frac{\log[2\pi\sigma^2(q_1-q_0)]}{2} - \frac{q_0}{2[q_1-q_0]},\end{aligned} \tag{15}$$

and where $\bar{\mathcal{J}}(\hat{\mathbf{x}})$ admits the following integral representation:

$$\begin{aligned}\bar{\mathcal{J}}(\hat{\mathbf{x}}) =& \int \frac{du_1 du_2}{2\pi\sqrt{\hat{q}_0\hat{\xi}_0 - \hat{z}^2}} \exp\left[\frac{\hat{\xi}_0 u_1^2 + \hat{q}_0 u_2^2 - 2\hat{z}u_1u_2}{2(\hat{q}_0\hat{\xi}_0 - \hat{z}^2)}\right] \times \\ &\times \log\left[e^{-\hat{\phi}}\sqrt{\frac{\pi}{2}}\frac{1}{\sqrt{2\hat{q}_1 - \hat{q}_0}} e^{\frac{(u_1-\hat{m})^2}{2(2\hat{q}_1-\hat{q}_0)}} \text{Erfc}\left(\frac{\hat{m}-u_1}{\sqrt{2(2\hat{q}_1-\hat{q}_0)}}\right) + \sqrt{\frac{\pi}{2}}\frac{1}{\sqrt{2\hat{\xi}_1 - \hat{\xi}_0}} e^{\frac{(u_2-\hat{p})^2}{2(2\hat{\xi}_1-\hat{\xi}_0)}} \text{Erfc}\left(\frac{-[\hat{p}-u_2]}{\sqrt{2(2\hat{\xi}_1-\hat{\xi}_0)}}\right)\right],\end{aligned} \tag{16}$$

derived under the assumptions:

$$2\hat{q}_1 - \hat{q}_0 > 0, \quad 2\hat{\xi}_1 - \hat{\xi}_0 > 0, \quad \hat{q}_0 < 0 \quad \hat{\xi}_0 < 0, \quad \hat{q}_0\hat{\xi}_0 - \hat{z}^2 > 0. \tag{17}$$

The saddle point equations fixing the values of the order and conjugate parameters can be obtained taking the derivatives of these expressions, as we recall below. Once the saddle point values are determined by solving the appropriate system of equations, plugging the resulting values into $\mathcal{A}_1$ and $\bar{\mathcal{A}}$ one obtaines the expression for the annealed and quenched complexity, respectively.

The variational problem and the self-consistent equations. Given the explicit form of the functionals $\mathcal{A}_1$ and $\bar{\mathcal{A}}$, the last step to obtain the complexity is to determine the values $\mathbf{x}_\star, \hat{\mathbf{x}}_\star$ of the order and conjugate parameters that solve the stationarity conditions

$$\frac{\delta\,\bar{\mathcal{A}}(\mathbf{x},\hat{\mathbf{x}},\phi)}{\delta\,\mathbf{x}}\Big|_{\mathbf{x}_\star,\hat{\mathbf{x}}_\star} = 0 = \frac{\delta\,\bar{\mathcal{A}}(\mathbf{x},\hat{\mathbf{x}},\phi)}{\delta\,\hat{\mathbf{x}}}\Big|_{\mathbf{x}_\star,\hat{\mathbf{x}}_\star}, \tag{18}$$

as well as the values $\mathbf{x}_\star^{(1)}, \hat{\mathbf{x}}_\star^{(1)}$ that optimize $\mathcal{A}_1$. In the quenched case, taking the variation of $\bar{\mathcal{A}}(\mathbf{x},\hat{\mathbf{x}},\phi)$ with respect to the 15 order and conjugate parameters we obtain two sets of equations of the form $\mathbf{x} = F_1[\hat{\mathbf{x}}]$ and $\hat{\mathbf{x}} = F_2[\mathbf{x}]$, respectively. These equations couple the 7 order parameters $\mathbf{x}$ with the 8 conjugate parameters $\hat{\mathbf{x}}$: inverting one of these sets, one can express the order parameters as a function of the conjugate parameters, $\mathbf{x} = f_3[\hat{\mathbf{x}}]$. The latter can then be fixed by solving the set of coupled self-consistent equations $\hat{\mathbf{x}} = F_2[f_3[\hat{\mathbf{x}}]]$: once the self-consistent values of the conjugate parameters $\hat{\mathbf{x}}$ are found, the order parameters can be determined and the quenched complexity can be obtained computing the action $\bar{\mathcal{A}}$ at the corresponding values of parameters. The annealed calculation is formally analogous. This scheme can be implemented for generic values of γ. A detailed discussion of the structure of the self-consistent equations and of the strategy to solve them can be found in [1].

On the unbounded phase. While the quenched complexity $\Sigma(\phi)$ is independent of μ, the typical properties of the equilibria (given by the saddle-point values of the parameters m, q_1, q_0) change with μ; in particular, decreasing μ at fixed σ, ϕ one finds that the solutions to the self-consistent equations m^*, q_1^*, q_0^* all increase and the system is driven towards the unbounded phase, signalled by a divergence of these parameters [4–7]. Given that we have access to the distribution of equilibria as a function of diversity, for each σ we can define a $\mu_c(\phi)$ such that for $\mu < \mu_c$ the system is in the unbounded phase. This curves is monotonically decreasing with ϕ, see Fig. 1. This suggests to define the boundary of the bounded phase in the σ, μ diagram thorough $\mu^* = \max_{\phi:\Sigma(\phi)\geq 0}\mu_c(\phi) = \mu_c(\phi_{\rm a})$, to ensure that *none* of the equilibria is in the unbounded phase, no matter their diversity. We remark that the unbounded phase defined in this way has a larger extension with respect to that estimated via the cavity approximation, since $\mu^* > \mu_c(\phi_{\rm cav})$. On the other hand, for $\mu = \mu^*$ the most numerous equilibria having $\phi = \phi_{\rm Max}$ are still in the bounded phase, so the phase boundary obtained using typical equilibria is yet different.

On the vanishing of the total complexity. We claimed in the main text that the total complexity $\Sigma_{\rm tot} = \Sigma(\phi_{\rm max})$ vanishes as $\Sigma_{\rm tot} \sim (\sigma - \sigma_c)^2$ as $\sigma \to \sigma_c^+$ for $\gamma = 0$, and that we expect this behavior to extend to $\gamma \neq 0$ *provided that* the maximum of $\Sigma(\phi)$ in the vicinity of σ_c lies in a region of ϕ in which the annealed calculation is correct. On the other hand, if at the maximum of $\Sigma(\phi)$ the quenched formalism has to be employed, we have indications of the fact that the exponent controlling the vanishing of the complexity is a different one. We motivate these claims in this subsection, and refer to Ref. [1] for the details. The total variation of $\Sigma_{\rm tot}$ with respect to σ is given by:

$$\frac{d\Sigma_{\rm tot}}{d\sigma} = \partial_\sigma\bar{\mathcal{A}}(\mathbf{x},\hat{\mathbf{x}},\phi)\Big|_{\mathbf{x}^*,\hat{\mathbf{x}}^*,\phi_{\rm max}} = \partial_\sigma\bar{\surd}(\mathbf{x})\Big|_{\mathbf{x}^*,\hat{\mathbf{x}}^*,\phi_{\rm max}} + \partial_\sigma\lceil(\phi)\Big|_{\mathbf{x}^*,\hat{\mathbf{x}}^*,\phi_{\rm max}}, \tag{19}$$

where we used the fact that $(\mathbf{x}^*, \hat{\mathbf{x}}^*, \phi_{\rm max})$ are a stationary point of $\bar{\mathcal{A}}(\mathbf{x},\hat{\mathbf{x}},\phi)$. For $\sigma < \sigma_c = \sqrt{2}(1+\gamma)^{-1}$, the system is in the unique equilibrium phase and a single, stable equilibrium exists. Its properties (described by the

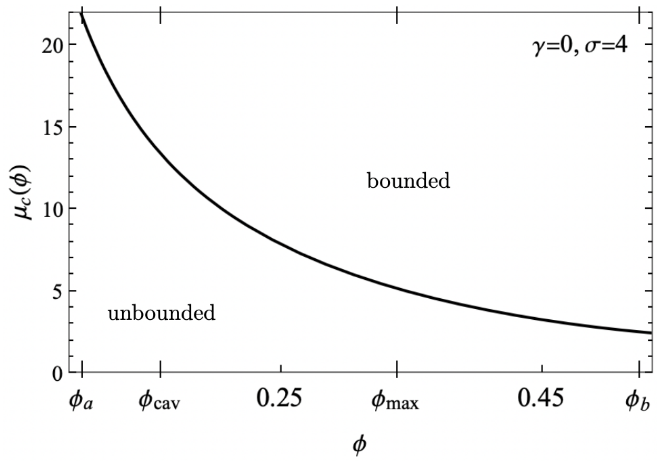

FIG. 1: Curve separating the unbounded ($\mu < \mu_c$) from the bounded ($\mu > \mu_c$) phase as a function of the diversity ϕ.

order parameters m, q_1) can be derived using the cavity method. For general γ and $\kappa = 1$, one finds [1] that at σ_c the equilibrium satisfies $m = \mu^{-1} = -(1+\gamma)p$, $q_1 = (1+\gamma)^2\xi_1$ and $\phi = \phi_{\max} = \phi_{\rm May} = [\sigma(1+\gamma)]^{-2}$. This implies:

$$\partial_\sigma \lceil(\phi)\Big|_{\sigma_c,\phi_{\max}} = -\frac{\gamma(1+\gamma)}{2\sqrt{2}}. \tag{20}$$

In order for the complexity to vanish quadratically at σ_c, this term should be compensated by the one obtained deriving the distribution of the forces $\bar{\surd}(\mathbf{x})$. If for $\sigma > \sigma_c$ and $\phi = \phi_{\max}$ the annealed calculation is exact, than one can replace $\bar{\surd}(\mathbf{x}) \to \surd_1(\mathbf{x})$, and use that for the values of parameters predicted by the cavity approximation it holds:

$$\partial_\sigma \surd_1\Big|_{\sigma_c,\phi_{\max}} = \frac{\gamma(1+\gamma)}{2\sqrt{2}}, \tag{21}$$

which cancels exactly (20). Therefore, if $\Sigma_{\rm tot}$ is analytic at σ_c, it has to vanish quadratically (one can check that the second derivative is not vanishing at the critical point). On the other hand, for $\gamma = 0$ we know that at $\phi_{\max}$ the annealed calculation is never correct, for any $\sigma > \sigma_c$. Assuming that this is still true for $\gamma = 0$, imposing that (19) vanishes and using the conditions given by the cavity approximation (in addition to $q_0 = (1+\gamma)^2\xi_0$ by symmetry) we obtain the following conditions for the order parameters:

$$\frac{z}{(1+\gamma)(q_1-q_0)^2}\left(\frac{\gamma z(q_1+q_0)}{2(q_1-q_0)} + q_0\right) = 0, \tag{22}$$

which implies either $z = 0$, or $z = 2q_0(q_1-q_0)/[\gamma(q_1+q_0)]$. Both these solutions however can be shown to be incompatible with the quenched self-consistent equations for this order parameter [1] except for the case $\gamma = 0$, when in fact it holds $z = 0$ at the transition point. Therefore, if for $\gamma \neq 0$ the total complexity at $\sigma \sim \sigma_c^+$ is quenched, one should expect a different power law since the linear contribution is not vanishing. We remark that the symmetric case $\gamma = 1$ is special, since the total complexity should vanish in a non-analytic way at the transition, due to the square root term in (12) whose argument vanishes when $\phi = \phi_{\rm May}, \sigma = \sigma_c$.

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